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EXTERNAL RADIATION FIELD IN RAYLEIGH–CABANNES ATMOSPHERES WITH CONSTANT AND LINEAR SOURCES

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Abstract. The external field of radiation in Rayleigh–Cabannes atmospheres with constant and linear sources is found using the resolvent matrix approach. If the internal sources are constant the external field may be described by the **X**-, **Y**-, and **H**-matrices. For the case with linear sources we need the derivatives of these matrices with respect to angular variable. The respective scheme for their determination is given.

A set of integro-differential equations for the **X**- and **Y**-matrices is derived and solved numerically. Some relations between the moments of the **H**-matrix are given and a sample of results for external fields are provided.

1. Introduction

In this paper we find the external fields of radiation for homogeneous plane-parallel Rayleigh–Cabannes atmospheres with constant or linear axially-symmetric internal sources using the resolvent matrix approch. The respective solution for the scalar case may be expressed in the X-, Y-, and H-functions and their derivatives with respect to angular variable depending on whether the atmosphere is optically finite or semi-infinite (Viik *et al.*, 1985). For the vector case we encounter a remarkable similarity, only in this time we have to do with respective matrices. As a result we have to find a way to obtain the numerical values of these matrices. For finite atmospheres we may use a procedure of determining the **X**- and **Y**-matrices based on the method of discrete ordinates (Viik, 1991). In the present paper we propose another way, namely closely following the respective derivation in scalar case we derive a set of matrix integro-differential equations which may be solved by discretizing the integral terms. For a semi-infinite atmosphere the spectrum of methods to find the **H**-matrix is much broader. The nonlinear integral equation for the **H**-matrix for pure Rayleigh scattering was solved iteratively first by Lenoble (1970) and Abhyankar and Fymat (1970, 1971). For the case of almost conservative scattering, which is most interesting, this method does not converge. Tables of the **H**-matrix for molecular (or Rayleigh–Cabannes) scattering have been given by Bond and Siewert (1971). They have used the Gauss–Seidel iteration which converges even for the most difficult case – conservative Rayleigh scattering but is rather time-consuming. Kriese and Siewert (1971) have proposed a rapidly converging scheme which incorporates the linear constaint thus uniquely specifying the solution (Pahor, 1968).

Lately, de Rooij *et al.* (1989) have elaborated a general iteration method to find the **H**-matrix which compares favourable with that of Kriese and Siewert (1971) since in the framework of this approach there is no need to solve the characteristic equation and

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to find the eigenvectors. Quite independently this same scheme has been rediscovered by Ivanov (1991).

Last but not least we may find the **H**-matrix by using the method of discrete ordinates (Viik, 1991).

2. The Equation of Transfer

The equation of transfer for the axially-symmetric part of the radiation field in a homogeneous plane-parallel atmosphere with internal sources may be written (Chandrasekhar, 1960; Domke, 1972) as

$$\mu \frac{\partial \mathbf{I}(\tau, \mu, \tau_0)}{\partial \tau} = \mathbf{I}(\tau, \mu, \tau_0) - \mathbf{A}_1(\mu)\mathbf{s}(\tau, \tau_0), \tag{1}$$

where

$$\mathbf{s}(\tau, \tau_0) = \frac{1}{2} \lambda \int_{-1}^{+1} \mathbf{A}_2(\mu')\mathbf{I}(\tau, \mu', \tau_0)\, d\mu' + \mathbf{s}_0(\tau). \tag{2}$$

In Equations (1) and (2) **I** is the intensity vector in the (l, r) representation; τ, the optical depth measured from the upper boundary of the atmosphere; μ, the cosine of the angle between the direction of travel of a photon and the positive τ-axis, τ_0 is the optical thickness of the atmosphere $(0 < \tau_0 \le \infty)$, λ is the albedo of the single scattering $(0 < \lambda \le 1)$ and $\mathbf{s}_0(\tau)$ is the Stokes's vector of the internal sources of radiative energy.

In the case of molecular scattering in the atmosphere the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are def ined in the way

$$\mathbf{A}_1(\mu) = \begin{pmatrix} 1 - \mu^2 & \mu^2 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

$$\mathbf{A}_2(\mu) = \mathbf{B}\mathbf{A}_1^T(\mu), \tag{4}$$

$$\mathbf{B} = \frac{3}{4} c \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}) + \frac{1}{2}(1 + c) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{5}$$

and

$$c = \frac{2(1 - \rho_n)}{2 + \rho_n},$$

where ρ_n is de depolarization factor $(0 \le c \le 1)$. The boundary conditions for Equation (1) are

$$\mathbf{I}(0, -\mu, \tau_0) = \mathbf{I}(\tau_0, \mu, \tau_0) = 0. \tag{6}$$

The solution of Equation (1) is

$$\mathbf{I}(\tau, \mu, \tau_0) = \mathbf{A}_1(\mu) \int_{\tau}^{\tau_0} \mathbf{s}(t, \tau_0) \exp(t - \tau)/\mu) \, \mathrm{d}t\mu \, , \tag{7}$$

$$\mathbf{I}(\tau, -\mu, \tau_0) = \mathbf{A}_1(\mu) \int_{0}^{\tau} \mathbf{s}(t, \tau_0) \exp(-(\tau - t)/\mu) \, \mathrm{d}t/\mu \, . \tag{8}$$

If we substitute Equations (7) and (8) into Equation (2) we obtain a matrix integral equation for the source function

$$\mathbf{S}(\tau, \tau_0) = \frac{1}{2} \lambda \int_{0}^{\tau_0} \mathbf{K}(|\tau - t|) \mathbf{s}(t, \tau_0) \, \mathrm{d}t + \mathbf{s}_0(\tau) \, , \tag{9}$$

where the kernel matrix is

$$\mathbf{K}(\tau) = \int_{0}^{1} \exp(-\tau/\mu) \Psi(\mu) \, \mathrm{d}\mu/\mu \tag{10}$$

and

$$\Psi(\mu) = \mathbf{A}_1(\mu) \mathbf{A}_2(\mu) \, . \tag{11}$$

The solution of Equation (9) may be written as

$$\Gamma(\tau, \tau', \tau_0) = \frac{1}{2} \lambda \int_{0}^{\tau_0} \mathbf{K}(|\tau - t|) \Gamma(t, r', \tau_0) \, \mathrm{d}t + \frac{1}{2} \lambda \mathbf{K}(|\tau - \tau'|) \, , \tag{12}$$

where Γ is the resolvent matrix of Equation (9). Hence, we have

$$\mathbf{s}(\tau, \tau_0) = \mathbf{s}_0(\tau) + \int_{0}^{\tau_0} \Gamma(\tau, t, \tau_0) \mathbf{s}_0(t) \, \mathrm{d}t \, . \tag{13}$$

In the following we shall need some properties of the resolvent matrix. According to Domke (1972, cf. Sobolev, 1972) it may be shown that

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau'}\right) \Gamma(\tau, \tau', \tau_0) = \Phi(\tau, \tau_0)^{+}(\tau', \tau_0) - \Phi(\tau_0 - \tau, \tau_0) \Phi^{+}(\tau_0 - \tau, \tau_0) \, , \tag{14}$$

where

$$\mathbf{D}^{+} = \mathbf{B} \mathbf{D}^{T} \mathbf{B}^{-1} \tag{15}$$

and

$$\Phi(\tau, \tau_0) = \Gamma(\tau, 0, \tau_0) . \tag{16}$$

The following relation of symmetry is valid

$$\Gamma(\tau, \tau', \tau_0) = \Gamma^+(\tau', \tau, \tau_0) . \tag{17}$$

If we use Equation (12) we may prove that

$$\Gamma(\tau, \tau', \tau_0) = \Gamma(\tau_0 - \tau, \tau_0 - \tau', \tau_0) . \tag{18}$$

Next we define the special matrices which are straightforward generalizations of the respective functions in scalar transfer: namely,

$$\mathbf{X}(\mu, \tau_0) = \mathbf{E} + \int_0^{\tau_0} \exp(-t/\mu)\Phi^+(t, \tau_0)\,dt , \tag{19}$$

$$\mathbf{Y}(\mu, \tau_0) = \mathbf{E}\exp(-\tau_0/\mu) + \int_0^{\tau_0} \exp(-t/\mu)\Phi^+(\tau_0 - t, \tau_0)\,dt \tag{20}$$

and in the case of a semi-infinite atmosphere

$$\mathbf{H}(\mu) = \mathbf{E} + \int_0^{\infty} \exp(-t/\mu)\Phi^+(t)\,dt . \tag{21}$$

In Equations (19) and (20) **E** is the identity matrix.

From Equations (19) and (20) it is easy to obtain a relation between the **X**- and **Y**-matrices,

$$\mathbf{Y}(\mu, \tau_0) = \mathbf{X}(-\mu, \tau_0)\exp(-\tau_0/\mu) . \tag{22}$$

3. Constant Sources

The formulas of the preceding section are valid if the albedo of single scattering is constant throughout the atmosphere but the internal sources of radiative energy may depend on the optical depth. Next we are going to confine ourselves to the case of constant internal sources since quite often this case serves as an idealized example for the transfer problems in spectral lines.

Let us describe the rate of energy output of constant sources by the Stokes's vector $\mathbf{s}_0$. According to Equations (13), (18)–(20) the values of the **s**-vector at the boundaries are

$$\mathbf{s}(0, \tau_0) = \mathbf{s}(\tau_0, \tau_0) = \mathbf{X}(\infty, \tau_0)\mathbf{s}_0 , \tag{23}$$

since

$$\mathbf{X}(\infty, \tau_0) = \mathbf{Y}(\infty, \tau_0) . \tag{24}$$

For a semi-infinite atmosphere Equation (23) becomes

$$\mathbf{s}(0) = \mathbf{H}(\infty)\mathbf{s}_0 . \tag{25}$$

It is evident that the problem with constant internal sources in a semi-infinite atmosphere has a solution only if $\lambda \neq 1$ since

$$\lim_{\lambda \to 1} \mathbf{H}(\infty) \to \infty .$$

As a next step we shall find the intensities of the emergent radiation. If we substitute Equation (13) into Equations (7) and (8) and by use of Equation (14) for partially integrating in Equations (7) and (8) we obtain

$$\mathbf{I}(0, \mu, \tau_0) = \mathbf{I}(\tau_0, -\mu, \tau_0) = \mathbf{A}_1(\mu)\,[\mathbf{X}^+(\mu, \tau_0) - \mathbf{Y}^+(\mu, \tau_0)]\mathbf{X}(\infty, \tau_0)\mathbf{s}_0 . \tag{26}$$

The intensities of the emergent radiation both at $\tau = 0$ and $\tau = \tau_0$ are equal as they should since the radiation field in the atmosphere is symmetrical with respect to the midplane $\tau = \tau_0/2$ as

$$\mathbf{I}(\tau, \mu, \tau_0) = \mathbf{I}(\tau_0 - \tau, -\mu, \tau_0) . \tag{27}$$

In the case of a semi-infinite atmosphere we have

$$\mathbf{I}(0, \mu) = \mathbf{A}_1(\mu)\mathbf{H}^+(\mu)\mathbf{H}(\infty)\mathbf{s}_0 , \tag{28}$$

since

$$\lim_{\tau_0 \to \infty} \mathbf{X}(\mu, \tau_0) = \mathbf{H}(\mu)$$

and

$$\lim_{\tau_0 \to \infty} \mathbf{Y}(\mu, \tau_0) = 0 ,$$

where we may add that the complete radiation field in a Rayleigh–Cabannes atmosphere with constant internal sources has been calculated by Viik (1990a) using the method of discrete ordinates.

4. Linear Sources

If we have linear internal sources in the atmosphere with the energy output rate $\mathbf{s}_1 \tau$ then according to Equations (13), (18)–(20) the source vectors at the boundaries are

$$\mathbf{s}(0, \tau_0) = \mathbf{Q}(\tau_0)\mathbf{s}_1 , \tag{29}$$

$$\mathbf{s}(\tau_0, \tau_0) = \mathbf{P}(\tau_0)\mathbf{s}_1 , \tag{30}$$

where

$$\mathbf{Q}(\tau_0) = \lim_{\mu \to \infty} \left[\mu^2 \frac{\partial \mathbf{X}(\mu, \tau_0)}{\partial \mu}\right], \tag{31}$$

$$\mathbf{P}(\tau_0) = \lim_{\mu \to \infty} \left[\mu^2 \frac{\partial \mathbf{Y}(\mu, \tau_0)}{\partial \mu}\right]. \tag{32}$$

We may show that

$$\mathbf{Q}(\tau_0) + \mathbf{P}(\tau_0) = \tau_0 \mathbf{X}(\infty, \tau_0). \tag{33}$$

For a semi-infinite atmosphere Equations (29) and (31) give

$$\mathbf{s}(0) = \mathbf{Q}\mathbf{s}_1 , \tag{34}$$

where

$$\mathbf{Q} = \lim_{\mu \to \infty} \left[\mu^2 \frac{\partial \mathbf{H}(\mu)}{\partial \mu}\right]. \tag{35}$$

If we substitute Equation (13) into Equations (7) and (8) and using Equations (14), (16)–(22) we obtain after a series of partial integrations

$$\mathbf{I}(0, \mu, \tau_0) = \mathbf{A}_1(\mu) \{\mathbf{X}^+(\mu, \tau_0)\mathbf{Q}(\tau_0) - \mathbf{Y}^+(\mu, \tau_0)\mathbf{P}(\tau_0) + \\ + \mu[\mathbf{X}^+(\mu, \tau_0) - \mathbf{Y}^+(\mu, \tau_0)]\mathbf{X}(\infty, \tau_0)\}\mathbf{s}_1 , \tag{36}$$

$$\mathbf{I}(\tau_0, -\mu, \tau_0) = \mathbf{A}_1(\mu) \{\mathbf{X}^+(\mu, \tau_0)\mathbf{P}(\tau_0) - \mathbf{Y}^+(\mu, \tau_0)\mathbf{Q}(\tau_0) - \\ - \mu[\mathbf{X}^+(\mu, \tau_0) - \mathbf{Y}^+(\mu, \tau_0)]\mathbf{X}(\infty, \tau_0)\}\mathbf{s}_1 . \tag{37}$$

In Section 6 we show how to ind the **Q**- and **P**-matrices. For a semi-infinite atmosphere Equation (36) gives

$$\mathbf{I}(0, \mu) = \mathbf{A}_1(\mu)\mathbf{H}^+(\mu)\,[\mathbf{Q} + \mu\mathbf{H}^+(\infty)]\mathbf{s}_1 . \tag{38}$$

All the formulas given in this section were numerically checked by exploiting the method of discrete ordinates. The package of FORTRAN subroutines written for an atmosphere with constant sources after having changed the block of determining the particular solution is applicable to the problem with linear internal sources as well.

5. The X-, Y-, and H-Matrices

In determining the external field of radiation for the case under consideration (and not only for that) we need the **X**-, **Y**-, and **H**-matrices. The determination of the first two and some of the properties of the **H**-matrix will be given in this section.

Equally with the well-known nonlinear integral equation for the **H**-matrix (Domke,

1972) ~~we obtain~~

$$\mathbf{H}(\mu) = \mathbf{E} + \frac{1}{2}\lambda\mu \int_0^1 \Psi(\mu') \frac{\mathbf{H}^+(\mu')\mathbf{H}(\mu)}{\mu + \mu'} \, \mathrm{d}\mu' , \tag{39}$$

there exists a linear integral equation (Mullikin, 1966; Domke, 1972) as

$$\mathbf{H}(\mu)\mathbf{T}(\mu) = \mathbf{E} + \frac{1}{2}\lambda\mu \int_0^1 \frac{\mathbf{H}(\mu')\Psi(\mu')}{\mu' - \mu} \, \mathrm{d}\mu' , \tag{40}$$

where

$$\mathbf{T}(z) = \mathbf{E} + \frac{1}{2}\lambda z \int_{-1}^1 \frac{\Psi(\mu')}{\mu' - z} \, \mathrm{d}\mu' , \quad z \ni [-1, 1] . \tag{41}$$

The matrix $\mathbf{T}$ is defined in the complex plane with the cut $[-1, 1]$. Extending the $\mathbf{H}$-matrix in the complex plane we observe that

$$\mathbf{H}^{-1}(z) = \mathbf{E} - \frac{1}{2}\lambda z \int_0^1 \frac{\Psi(\mu')\mathbf{H}^+(\mu')}{z + \mu'} \, \mathrm{d}\mu' . \tag{42}$$

From Equations (40) and (42) follows that

$$\mathbf{H}(z)\mathbf{T}(z) = [\mathbf{H}^+(-z)]^{-1} , \tag{43}$$

which is very similar to the respective result in the scalar theory.

Substituting Equations (41) and (42) into (43) and expanding the result in exponents of z^{-1}, Domke (1972) has obtained a general formula which connects the moments of the $\mathbf{H}$-matrix. Here we are interested only in that of the lowest order

$$\mathbf{E} - \lambda\Psi_0 = (\mathbf{E} - \tfrac{1}{2}\lambda\mathbf{h}_0^+)(\mathbf{E} - \tfrac{1}{2}\lambda\mathbf{h}_0) , \tag{44}$$

where

$$\mathbf{h}_0 = \int_0^1 \mathbf{H}(\mu)\Psi(\mu) \, \mathrm{d}\mu$$

and

$$\Psi_0 = \int_0^1 \Psi(\mu) \, \mathrm{d}\mu .$$

For the scalar case it is possible to find the zeroth moment of the H-function explicitly but for the matrix case we can find only some relations between the moments insufficient to explicitly define them.

We may point out a way to obtain another set of relations between the moments of the **H**-matrix by exploiting the quadratic integrals of matrix transfer by Ivanov (1990) (cf. Rybicki, 1977). Ivanov has found that if the internal sources of radiation in a semi-infinite atmosphere are not polarized

$$s_{0l} = s_{0r} = \tfrac{1}{2}(1 - \lambda),$$

then in the (l, r)-representation the following statement is valid

$$4[s_l^2(0) + 2s_r^2(0)] = 3(1 - \lambda). \tag{45}$$

By use of Equation (25) we obtain from (45)

$$[H_{11}(\infty) + H_{12}(\infty)]^2 + 2[H_{21}(\infty) + H_{22}(\infty)]^2 = 3((1 - \lambda)^{-1}. \tag{46}$$

Ivanov has considered a strange but nevertheless rather fruitful non-physical situation with the internal sources

$$t_{0l} = \frac{2}{\sqrt{8}}\,(1 - \tfrac{7}{10}c\lambda)$$

and

$$t_{0r} = -\frac{1}{\sqrt{8}}\,(1 - \tfrac{7}{10}c\lambda),$$

where symbol t is used instead of s to distinguish this problem. For this case Ivanov has obtained that

$$4[t_l^2(0) + 2t_r^2(0)] = 3(1 - \tfrac{7}{10}c\lambda). \tag{47}$$

After substitution we have

$$[2H_{11}(\infty) - H_{12}(\infty)]^2 + 2[2H_{21}(\infty) - H_{22}(\infty)]^2 = 6(1 - \tfrac{7}{10}c\lambda)^{-1}. \tag{48}$$

The third relation may be obtained from the result by Ivanov in the form

$$s_l(0)t_l(0) + s_r(0)t_r(0) = 0. \tag{49}$$

Again, after substitution we obtain

$$2H_{11}^2(\infty) - H_{12}^2(\infty) + 4H_{21}^2(\infty) - 2H_{22}^2(\infty) + H_{11}(\infty)H_{12}(\infty) + \\ + 2H_{21}(\infty)H_{22}(\infty) = 0. \tag{50}$$

The results (46), (48), and (50) may be evaluated in moments of the **H**-matrix since for $\mu \to \infty$ Equation (39) gives

$$\mathbf{H}(\infty) = (\mathbf{E} - \tfrac{1}{2}\lambda\mathbf{h}_0^+)^{-1}, \tag{51}$$

where

$$\mathbf{h}_k^+ = \int_0^1 \Psi(\mu)\mathbf{H}^+(\mu)\mu^k\,\mathrm{d}\mu, \quad k = 0, 1, 2, \ldots. \tag{52}$$

Next we show how Equations (19) and (20) may be used to derive a basic set of integro-differential equations for the X- and Y-matrices. The way of doing it is very similar to that of in the case of scalar transfer. Thus, if we differentiate Equations (19) and (20) with respect to parameter τ_0 we obtain

$$\frac{\partial \mathbf{X}(\mu, \tau_0)}{\partial \tau_0} = \mathbf{\Phi}^+(\tau_0, \tau_0)\mathbf{Y}(\mu, \tau_0), \tag{53}$$

$$\frac{\partial \mathbf{Y}(\mu, \tau_0)}{\partial \tau_0} = -\mathbf{Y}(\mu, \tau_0)/\mu + \mathbf{\Phi}^+(\tau_0, \tau_0)\mathbf{X}(\mu, \tau_0). \tag{54}$$

In deriving these equations we have used the Krein–Bellman formula

$$\frac{\partial \mathbf{\Phi}(\tau, \tau_0)}{\partial \tau_0} = \mathbf{\Phi}(\tau_0 - \tau, \tau_0)\mathbf{\Phi}(\tau_0, \tau_0). \tag{55}$$

It is possible to show that

$$\mathbf{\Phi}^+(\tau_0, \tau_0) = \frac{1}{2}\lambda \int_0^1 \mathbf{\Psi}(\mu)\mathbf{Y}^+(\mu, \tau_0)\, d\mu/\mu. \tag{56}$$

Thus we have derived a system of integro-differential equations which are easy to solve after substituting the integral in Equation (56) by a suitably chosen quadrature formula.

The respective boundary conditions may be found from Equations (19) and (20)

$$\mathbf{X}(\mu, 0) = \mathbf{Y}(\mu, 0) = \mathbf{E}. \tag{57}$$

We note that substituting the integral in Equation (56) by an approximate sum provides us only with the values of the X- and Y-matrices at the quadrature points. However, this difficulty may be overcome by using the nonlinear integral equations (Domke, 1972; Viik, 1991) in the form

$$\mathbf{X}(\mu, \tau_0) = \mathbf{E} + \frac{1}{2}\lambda\mu \int_0^1 \frac{\mathbf{\Psi}(\mu')}{\mu' + \mu} [\mathbf{X}^+(\mu', \tau_0)\mathbf{X}(\mu, \tau_0) - $$
$$ - \mathbf{Y}^+(\mu', \tau_0)\mathbf{Y}(\mu, \tau_0)]\, d\mu', \tag{58}$$

$$\mathbf{Y}(\mu', \tau_0) = \mathbf{E}\exp(-\tau_0/\mu) + \frac{1}{2}\lambda\mu \int_0^1 [\mathbf{Y}^+(\mu', \tau_0)\mathbf{X}(\mu, \tau_0) - $$
$$ - \mathbf{X}^+(\mu', \tau_0)\mathbf{Y}(\mu, \tau_0)]\, d\mu'. \tag{59}$$

In the matrix case it is evidently impossible to obtain simple relations between the moments of the X- and Y-matrices. This is explained by the non-commutativity of the relevant matrices. However, it is possible to find the values of of the X- and Y-matrices at $\mu \to \infty$. We obtain from Equations (58) and (59)

$$\mathbf{X}(\infty, \tau_0) = \mathbf{Y}(\infty, \tau_0) = \{\mathbf{E} - \tfrac{1}{2}\lambda[\mathbf{x}_0^+(\tau_0) - \mathbf{y}_0^+(\tau_0)]\}^{-1}, \tag{60}$$

where

$$\mathbf{x}_0^+(\tau_0) = \int_0^1 \Psi(\mu)\mathbf{X}^+(\mu, \tau_0)\mu^k \, \mathrm{d}\mu \tag{61}$$

and

$$\mathbf{y}_0^+(\tau_0) = \int_0^1 \Psi(\mu)\mathbf{Y}^+(\mu, \tau_0)\mu^k \, \mathrm{d}\mu , \quad k = 0, 1, 2, \ldots . \tag{62}$$

6. The Q- and P-Matrices

To evaluate the external radiation field for the atmosphere with linear interval sources we need the **Q**- and **P**-matrices. In the following we show how to find them.

We may write Equation (39) in the form

$$\mathbf{H}^{-1}(\mu) = \mathbf{E} - \frac{1}{2}\lambda\mathbf{h}_0^+ + \frac{1}{2}\lambda \int_0^1 \frac{\Psi(\mu')\mathbf{H}^+(\mu')}{\mu + \mu'} \mu' \, \mathrm{d}\mu' . \tag{63}$$

If e differentiate this equation with respect to μ we obtain

$$\frac{\partial \mathbf{H}(\mu)}{\partial \mu} = \frac{1}{2}\lambda\mathbf{H}(\mu) \int_0^1 \frac{\Psi\lambda(\mu')\mathbf{H}^+(\mu')}{(\mu + \mu')^2} \mu' \, \mathrm{d}\mu' \mathbf{H}(\mu), \tag{64}$$

since

$$\frac{\partial \mathbf{H}^{-1}(\mu)}{\partial \mu} = -\mathbf{H}^{-1}(\mu) \frac{\partial \mathbf{H}(\mu)}{\partial \mu} \mathbf{H}^{-1}(\mu).$$

Thus, according to Equations (52) and (64) we have

$$\mathbf{Q} = \tfrac{1}{2}\lambda\mathbf{H}(\infty)\mathbf{h}_1^+\mathbf{H}(\infty). \tag{65}$$

The respective equations for the $\mathbf{Q}(\tau_0)$-nd $\mathbf{P}(\tau_0)$-matrices are more complicated. First, we differentiate Equations (58) and (59) with respect to μ. This results in a set of coupled linear algebraic matrix equations which may simply be solved and the respective derives found. Since we are interested in determination of the **Q**- and **P**-matrices, we do not dwell on that point. If we multiply the differentiated equations by μ^2 and approaching the limit $\mu \to \infty$ we finally have

$$[\mathbf{E} - \tfrac{1}{2}\lambda \mathbf{x}_0^+(\tau_0)]\mathbf{Q}(\tau_0) + \tfrac{1}{2}\lambda \mathbf{y}_0^+(\tau_0)\mathbf{P}(\tau_0) = \\ = \tfrac{1}{2}\lambda[\mathbf{x}_1^+(\tau_0) - \mathbf{y}_1^+(\tau_0)]\mathbf{X}(\infty, \tau_0), \qquad (66)$$

$$\tfrac{1}{2}\lambda \mathbf{y}_0^+(\tau_0) + [\mathbf{E} - \tfrac{1}{2}\lambda \mathbf{x}_0^+(\tau_0)]\mathbf{P}(\tau_0) = \\ = \mathbf{E}\tau_0 + \tfrac{1}{2}\lambda[\mathbf{x}_1^+(\tau_0) - \mathbf{y}_1^+(\tau_0)]\mathbf{X}(\infty, \tau_0). \qquad (67)$$

This is a linear algebraic matrix system which can easily be solved if we take into account Equation (33). The solution is

$$\mathbf{Q}(\tau_0) = (\alpha - \beta)^{-1}[\gamma - \tau_0\beta\mathbf{X}(\infty, \tau_0)], \qquad (68)$$

$$\mathbf{P}(\tau_0) = (\alpha - \beta)^{-1}[\tau_0\alpha\mathbf{X}(\infty, \tau_0) - \gamma], \qquad (69)$$

where

$$\alpha = \mathbf{E} - \tfrac{1}{2}\lambda \mathbf{x}_0^+(\tau_0), \qquad \beta = \tfrac{1}{2}\lambda \mathbf{y}_0^+(\tau_0)$$

and

$$\gamma = \tfrac{1}{2}\lambda[\mathbf{x}_1^+(\tau_0) - \mathbf{y}_1^+(\tau_0)]\mathbf{X}(\infty, \tau_0).$$

This completes the determination of the external radiation field in the case of linear internal sources.

7. Numerical Solution and Some Results

First we note that Equations (53), (54), and (56) are very similar to respective scalar equations (Chandrasekhar, 1960) which have been solved numerically by Bellman *et al.* (1966). For discretizing the integral term they have chosen the Gauss–Legendre quadrature rule on the interval (0, 1). We have followed their approach and the result are of high accuracy, especially when exploiting the scheme by Bulirsch and Stoer (Press *et al.*, 1986). If the order of quadrature is N we have to solve $8N$ simultaneous ordinary differential equations. On a PC/AT 286/287 computer for $N = 16$, $\tau_0 = 5$, $\lambda = 1$, $c = 1$, $\Delta\tau_0 = 0.01$ and asking for accuracy of 10^{-8} the CPU time was 464 s. For the given set of parameters the fixed stepsize results in much shorter CPU time than the case with the adaptive scheme though the number of steps for the adaptive scheme is almost three imes smaller.

This example shows that this method for obtaining the X- and Y-matrices compares very unfavourably with the method described by Viik (1991). Nevertheless, there may arise situations where the only advantage of the given method – its simplicity – is substantial.

To check the accuracy of the X- and Y-matrices we have used Equations (58) and (59). In our calculations Equations (58) and (59) for the interval (0, 1) were satisfied to one part in 10^{15} (when keeping the stepsize of integration equal to 0.01).

Since the polarizing properties of the Rayleigh–Cabannes atmospheres with constant internal sources have been dealt with by Viik (1990a) then in the following we shall discuss some features of the extenral radiation field for the atmospheres with linear internal sources

$$s_{1l}(\tau) = s_{1r}(\tau) = \tau(1 - \lambda)/2 .$$

It is evident that the closer λ is to unity the less is the difference between the external fields of radiation for the atmospheres with constant and linear internal sources. Let us define the Rubenson degree of polarization (RDP) as

$$P(\tau, \mu, \tau_0) = \frac{I_r(\tau, \mu, \tau_0) - I_l(\tau, \mu, \tau_0)}{I_r(\tau, \mu, \tau_0) + I_l(\tau, \mu, \tau_0)} .$$

We are mostly interested in RDP at the limb, thus $\mu = 0$ and as far as we have confined ourselves to the external field, $\tau = 0$ or $\tau = \tau_0$.

In Figure 1 the values of $P(0, 0, \infty)$ as the functions of the depolarization factor c are presented. Between the curves labeled 1 ($\lambda = 10^{-6}$) and 4 ($\lambda = 1-10^{-6}$) lies practically

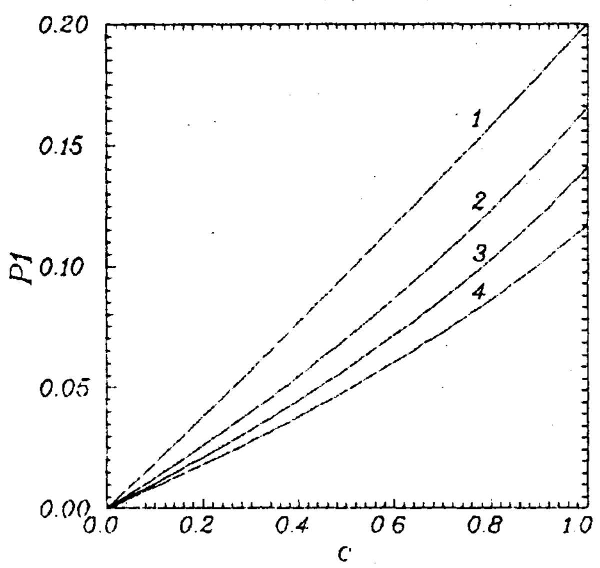

Fig. 1. $P(0, 0)$ as a function of the depolarization factor for a semi-infinite atmosphere with linear internal sources: $1 - \lambda = 10^{-6}$, $2 - \lambda = 0.7$, $3 - \lambda = 0.9$, $4 - 0.999999$.

the whole physical region of RDPs. It may be noted that $P(0, 0, \infty)$ for $c = 1.0$ and $\lambda \to 0$ approaches 0.2. This is one of the most pronounced differences between the external fields of radiation for the atmospheres with linear and constant internal sources since in the latter case $P(0, 0, \infty)$ approaches zero for every c if $\lambda \to 0$.

The dependence of $P(0, 0, \infty)$ on the albedo of single scattering is not very strong. For a pure Rayleigh-scattering atmosphere ($c = 1.0$) the RDP changes from the well-known Chandrasekhar–Sobolev limit 0.117129 (Viik, 1990b) at $\lambda \to 1$ to 0.2 at $\lambda \to 0$ (Figure 2).

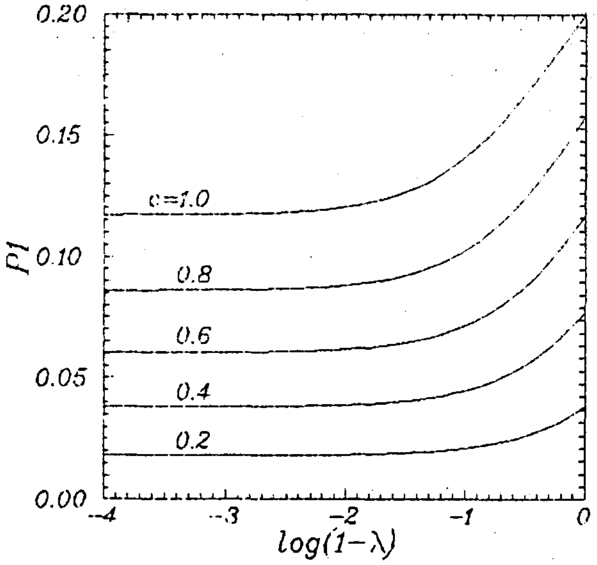

Fig. 2. $P(0, 0)$ as a function of the albedo of single scattering for a semi-infinite atmosphere with linear internal sources.

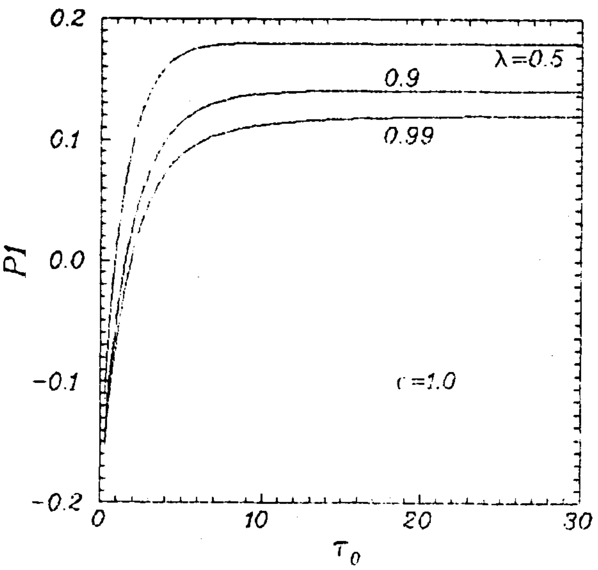

Fig. 3. $P(0, 0, \tau_0)$ as a function of the optical thickness for inite atmospheres with linear internal sources (c = 1.0).

The dependence of the RDP at the limb on the optical thickness for the emergent radiation at $\tau = 0$ is given in Figure 3. The RDP reaches the asymptotic level already at small optical thicknesses even for the albedos of single scattering close to unity.

The same dependence, only at $\tau = \tau_0$, is given in Figure 4. We observe that both

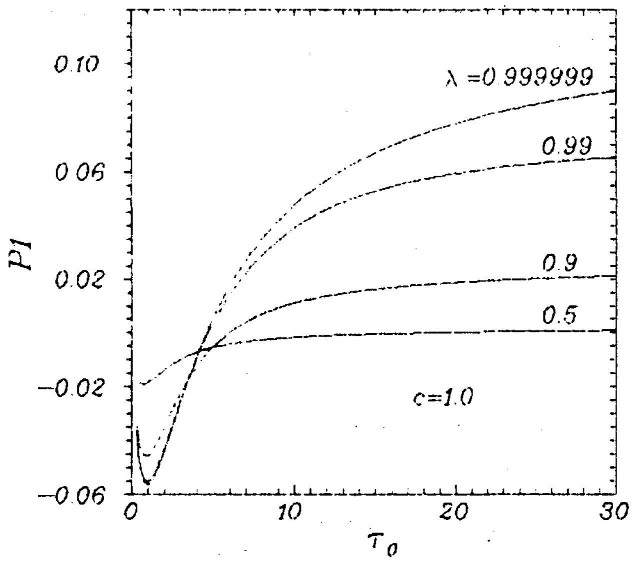

Fig. 4. $P(\tau_0, 0, \tau_0)$ as a function of the optical thickness for finite atmospheres with linear internal sources ($c = 1.0$).

$P(0, 0, \tau_0)$ and $P(\tau_0, 0, \tau_0)$ are negative for small optical thicknesses. Our calculations have shown that it is possible to find such an atmosphere with linear internal sources for which both $P(0, 0, \tau_0)$ and $P(\tau_0, 0, \tau_0)$ are zero simultaneously. This curious situation takes place for $\tau_0 \approx 3.37$ if $\lambda \to 1.133$. In this connection we would like to point out that the method of determining the **X**- and **Y**-matrices (and, respectively, the **Q**- and **P**-matrices) given in Sections 5 and 6 is powerful enough to cope with the problems with $\lambda > 1$.

From Figure 4 one sees that the distribution of $P(\tau_0, 0, \tau_0)$ has a minimum the location of which on the τ_0-axis depends weakly on λ, extending from $\tau_0 \approx 0.60$ at $\lambda \to 0$ to $\tau_0 \approx 0.97$ at $\lambda \to 1$. The same behaviour of polarization at the limb is encountered when considering finite atmospheres with constant sources. In this case the minima are shallower and they occur at optically thinner atmospheres (the respective region of optical thicknesses is from 0.31 to 0.67).

In Table I we give the values of the **Q***-matrix in 10 significant figures while

$$\mathbf{Q}^* = (1 - \lambda)\mathbf{Q}\,.$$

To guarantee this accuracy we have used the method of de Rooij *et al.* (1989) for obtaining the **H**-matrix while the order of quadrature was $N = 91$.

TABLE I

The Q*-matrix

λ	Q_{11}	Q_{12}	Q_{21}	Q_{22}
		$c = 0.20$		
0.100000	0.0077317911	0.0168060949	0.0060618809	0.0203371344
0.300000	0.0249539704	0.0543371383	0.0212422222	0.0635356237
0.500000	0.0457735667	0.0990685778	0.0419122941	0.1118815966
0.700000	0.0732820482	0.1559547917	0.0713861878	0.1700867158
0.900000	0.1179484424	0.2427627001	0.1204808957	0.2558351384
0.990000	0.1648109140	0.3308520195	0.1707144384	0.3435648617
0.999900	0.1853984679	0.3708122293	0.1922192952	0.3844631400
0.999999	0.1874765040	0.3749531643	0.1943751623	0.3887505736
		$c = 0.40$		
0.100000	0.0088716180	0.0144092753	0.0054523725	0.0215722764
0.300000	0.0276100636	0.0478739191	0.0195831747	0.0670685147
0.500000	0.0486684001	0.0897855942	0.0396555356	0.1172353202
0.700000	0.0747750661	0.1454827318	0.0693955192	0.1763368981
0.900000	0.1160001016	0.2330036295	0.1203701770	0.2613198009
0.990000	0.1603552182	0.3213531336	0.1725573411	0.3480265615
0.999900	0.1802986528	0.3606071319	0.1945259001	0.3890848758
0.999999	0.1823197632	0.3646396284	0.1967105237	0.3934213823
		$c = 0.60$		
0.100000	0.0100663937	0.0119427676	0.0048142901	0.0228433372
0.300000	0.0307173613	0.0408179656	0.0176685133	0.0709268670
0.500000	0.0526107528	0.0790389222	0.0367411359	0.1234487518
0.700000	0.0776692003	0.1327639823	0.0663877107	0.1840024182
0.900000	0.1147030354	0.2212361651	0.1195521452	0.2681067855
0.990000	0.1555206084	0.3106884573	0.1744704114	0.3530491644
0.999900	0.1746510691	0.3493025136	0.1970546504	0.3941555600
0.999999	0.1766082651	0.3532165385	0.1992721053	0.3985446784
		$c = 0.80$		
0.100000	0.0113189952	0.0094032865	0.0041461470	0.0241520073
0.300000	0.0343571579	0.0330749160	0.0154531460	0.0751622698
0.500000	0.0579710465	0.0663861032	0.0329459470	0.1307804677
0.700000	0.0828213774	0.1167084677	0.0617872992	0.1937704290
0.900000	0.1148575037	0.2061197913	0.1173560879	0.2771053435
0.990000	0.1502423914	0.2982762984	0.1763978140	0.3589413067
0.999900	0.1682335731	0.3364495228	0.1998901140	0.3998490657
0.999999	0.1701161651	0.3402321598	0.2021469966	0.4042946887
		$c = 1.00$		
0.100000	0.0126324891	0.0067873402	0.0034463592	0.0255000833
0.300000	0.0386297599	0.0245294643	0.0128815369	0.0798377235
0.500000	0.0652967231	0.0511866557	0.0279421630	0.1396041712
0.700000	0.0918323079	0.0953656453	0.0545361894	0.2068677965
0.900000	0.1185878388	0.1845452028	0.1120964683	0.2904407082
0.990000	0.1445551399	0.2828204610	0.1780973194	0.3664345224
0.999900	0.1606199386	0.3211815636	0.2031907313	0.4064962440
0.999999	0.1624084796	0.3248163842	0.2055000192	0.4110011969

8. Conclusions

We have shown that it is possible to find the external field of radiation for the Rayleigh–Cabannes homogeneous plane-parallel atmospheres with constant or linear internal sources using the resolvent matrix approach without knowing the resolvent matrix itself. This problem may be reduced to a mathematical problem of determination of the **X**-, **Y**-, and **H**-matrices and their derivatives with respect to angular variable.

The described approach renders the numerical results very accurate.

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