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A GENERALISED PRINCIPLE OF INVARIANCE AND RADIATION FIELD IN MULTILAYER ATMOSPHERES

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Abstract. A generalized principle of invariance is derived for a plane-parallel atmosphere. On the basis of this principle a method for determining the radiation field in a multilayer atmosphere is proposed. This method, the first part of which is the well-known adding method, permits the application to problems involving optically finite as well as semi-infinite atmospheres. The reflecting boundaries may be incorporated, though in that case it is not possible to use the adding method.

Some numerical results are given for the standard and Milne problems and for the problem with internal sources.

1. Introduction

Let us consider a finite plane-parallel inhomogeneous atmosphere of optical thickness τ_0 without internal sources in which both absorption and isotropic scattering take place.

The upper and lower surfaces of this atmosphere are illuminated while the intensities are $I_A(0, \mu)$ and $I_B(\tau_0, -\mu)$, respectively. The angle of reflection $\arccos \mu$ is measured with respect to the inward drawn normal. Cogley and Domanus (1972) have shown that there exists a relation

$$B(\tau, \tau_0) = \frac{2}{F}\int_0^1 [I_A(0, w)J(\tau, w, \tau_0) + I_B(\tau_0, -w)\bar{J}(\tau, w, \tau_0)]\, dw\,, \qquad (1)$$

where $B(\tau, \tau_0)$ is the source function for the problem considered, J is the source function for the standard problem, i.e., if the uniform parallel rays of net flux $\pi F w$ are incident on the top of a plane-parallel atmosphere. $\bar{J}$ is the source function for the reversed standard problem, i.e., the uniform parallel rays of the same net flux are incident on the bottom of a plane-parallel atmosphere. The acute angle between the inward drawn normal and the direction of the incident rays is $\arccos w$.

The relation of Cogley and Domanus is a starting point to obtaining many well-known principles of invariance as well as their generalizations for inhomogeneous atmospheres.

2. Principle of Invariance

Firstly, let us consider the standard problem for an atmosphere of optical thickness τ_0 consisting of three layers of optical thicknesses τ_1, τ_2, and τ_3. In this

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case it is possible to deduce the following formula for the second layer (as the most general layer in the sense of illumination)

$$
\begin{aligned}
I(\tau_1+t,\mu,\mu_0,\tau_0) = {} & I(t,\mu,\mu_0,\tau_2)\exp\left(-\frac{\tau_1}{\mu_0}\right) + \\
& + I(\tau_1,\mu,\mu_0,\tau_0)\exp\left(-\frac{t}{\mu}\right)\vartheta(\mu) \\
& + I(\tau_{12},\mu,\mu_0,\tau_0)\exp\left(\frac{\tau_2-t}{\mu}\right)\vartheta(-\mu) + \\
& + \frac{2}{F}\int_0^1 I(t,\mu,w,\tau_2)I(\tau_1,w,\mu_0,\tau_0)\,\mathrm{d}w + \\
& + \frac{2}{F}\int_0^1 \bar{I}(t,\mu,w,\tau_2)I(\tau_{12},-w,\mu_0,\tau_0)\,\mathrm{d}w, \qquad (2) \\
& -1 \leqslant \mu \leqslant 1,
\end{aligned}
$$

where

$$\tau_{12} = \tau_1 + \tau_2,$$

$$\vartheta(\mu) = \begin{cases} 1, & \mu > 0, \\ 0, & \mu < 0, \end{cases}$$

and the optical depth t is measured from the level $\tau = \tau_1$.

If the second layer is homogeneous, then

$$\bar{I}(t,\mu,\mu_0,\tau_2) = I(\tau_2 - t, -\mu, \mu_0, \tau_2).$$

From the relation (2) it is possible to derive the principles of invariance of Ambarzumian (1960), Chandrasekhar (1960), Ivanov (1975), Yengibarian-Mnatsakanian (1974), Ivanov (1976) and Yanovitskij (1979) for inhomogeneous atmospheres as well.

It is important to stress that relation (2) is valid also for the case when there are no incident rays and the radiation field in an atmosphere is created by the internal sources. In this case we have $\mu_0 \to \infty$ and

$$
\begin{aligned}
I(\tau_1+t,\mu,\tau_0) = {} & I(t,\mu,\tau_2) + I(\tau_1,\mu,\tau_0)\exp\left(-\frac{t}{\mu}\right)\vartheta(\mu) + \\
& + I(\tau_{12},\mu,\tau_0)\exp\left(\frac{\tau_2-t}{\mu}\right)\vartheta(-\mu) + \\
& + \frac{2}{F}\int_0^1 I(t,\mu,w,\tau_2)I(\tau_1,w,\tau_0)\,\mathrm{d}w + \\
& + \frac{2}{F}\int_0^1 \bar{I}(t,\mu,w,\tau_2)I(\tau_{12},-w,\tau_0)\,\mathrm{d}w, \qquad -1 \leqslant \mu \leqslant 1.
\end{aligned}
\qquad (3)
$$

much easier to start adding the layers from the bottom if the layers are inhomogeneous.

The last step of addition results in obtaining the emergent intensities from the whole compound atmosphere and the intensities at the first contact surface. In order to find the intensities at the other contact surfaces we make use of the two relations which follow from relation (2). For an atmosphere consisting of three layers we have

$$\begin{aligned} I(\tau_{12}, \mu, \mu_0, \tau_0) = {} & I(\tau_2, \mu, \mu_0, \tau_2)\exp\left(-\frac{\tau_1}{\mu_0}\right) + \\ & + I(\tau_1, \mu, \mu_0, \tau_0)\exp\left(-\frac{\tau_2}{\mu}\right) + \\ & + \frac{2}{F}\int_0^1 I(\tau_2, \mu, w, \tau_2) I(\tau_1, w, \mu_0, \tau_0)\, \mathrm{d}w + \\ & + \frac{2}{F}\int_0^1 \bar{I}(\tau_2, \mu, w, \tau_2) I(\tau_{12}, -w, \mu_0, \tau_0)\, \mathrm{d}w\,, \end{aligned} \tag{9}$$

$$\begin{aligned} I(\tau_{12}, -\mu, \mu_0, \tau_0) = {} & I(0, -\mu, \mu_0, \tau_0 - \tau_{12})\exp\left(-\frac{\tau_{12}}{\mu_0}\right) + \\ & + \frac{2}{F}\int_0^1 I(0, -\mu, w, \tau_0 - \tau_{12}) I(\tau_{12}, w, \mu_0, \tau_0)\, \mathrm{d}w\,. \end{aligned} \tag{10}$$

It is easy to see that in Equations (9) and (10) only intensities $I(\tau_{12}, \pm\mu, \mu_0, \tau_0)$ are unknown. They may be found by solving the integral equation. This is the point where we use the intensities we have stored in the forward sweep. This process may be continued downwards (backward sweep) until all the intensities on the contact surfaces are found. If we are interested in the intensities at arbitrary optical depth then these intensities may be found by using relation (2).

The backward sweep described in a paper by the author (1981) is more complicated and laborious. Being based on the use of Chandrasekhar principles only rather than relation (2) it does not permit application to semi-infinite atmospheres.

If the radiation field in a multi-layer atmosphere is created by internal sources the described approach is valid after the formal change $\mu_0 \to \infty$.

4. Radiation Field in a Semi-Infinite Atmosphere

The method proposed is applicable to a semi-infinite atmosphere if only the albedo of single scattering approaches a constant value λ_0 when $\tau \to \infty$. In this case it is possible to separate almost homogeneous semi-infinite atmosphere with the albedo of single scattering λ_0 and the rest of the atmosphere can be divided into optically thin layers which may be considered homogeneous. After such a division we may proceed as described above since the downward intensity at $\tau \to \infty$ is of no importance for this scheme.

$$I(0, \mu, \tau_0) = 2\int_0^1 \rho_A(\mu, w) I(0, -w, \tau_0) w \, dw, \tag{17}$$

$$I(\tau_0, -\mu, \tau_0) = 2\int_0^1 \rho_B(\mu, w) I(\tau_0, w, \tau_0) w \, dw, \tag{18}$$

$$\begin{aligned} I(\tau_0, \mu, \tau_0) = {} & I^*(\tau_0, \mu, \tau_0) + I(0, \mu, \tau_0)\exp\left(-\frac{\tau_0}{\mu}\right) + \\ & + \frac{2}{F}\int_0^1 I(\tau_0, \mu, w, \tau_0) I(0, w, \tau_0)\, dw + \\ & + \frac{2}{F}\int_0^1 I(0, -\mu, w, \tau_0) I(\tau_0, -w, \tau_0)\, dw, \end{aligned} \tag{19}$$

where I^* means the intensity in the same atmosphere without reflecting boundaries. System (16)–(19) is to be solved as a whole.

If the albedo of single scattering depends on the optical depth then the atmosphere is to be divided into optically thin layers which may be considered as homogeneous. In this case we have to write formula (3) for every contact surface.

6. Illustrations

As an illustration of the above method we have calculated the radiation field in finite and semi-infinite atmospheres where the albedo of single scattering is described by the following relation (Pomraning and Larsen, 1980)

$$\lambda(\tau) = \left[c + kb\exp\left(-\frac{\tau}{s}\right)\right]\left[1 + b\exp\left(-\frac{\tau}{s}\right)\right]^{-1} \tag{20}$$

with parameters $c = 0.96$, $k = 0.686954$, and $s = 2$.

In Figures 1 to 4 the upward intensities for different problems and different parameter b are plotted. Since the cases with $b = 0$ and $b = \infty$ correspond to the homogeneous atmospheres with $\lambda = c$ and $\lambda = k$, respectively, it is possible to follow the effects of inhomogeneity upon the upward intensity. For the atmosphere with internal source (Figure 2) we have chosen for simplicity

$$B_0(\tau) = \lambda(\tau).$$

In Figure 5 the intensities in a homogeneous atmosphere with internal sources are plotted. The atmosphere is bounded by the Lambert surfaces from the both sides. The comparison with the radiation field in the atmosphere without reflecting boundaries clearly indicates the amplifying effect of the boundaries.

All the calculations have been programmed in FORTRAN and performed on a Minsk 32 computer. In order to obtain the emergent intensities from separate layers we have used the kernel approximation method (Heinlo and Viik, 1978).

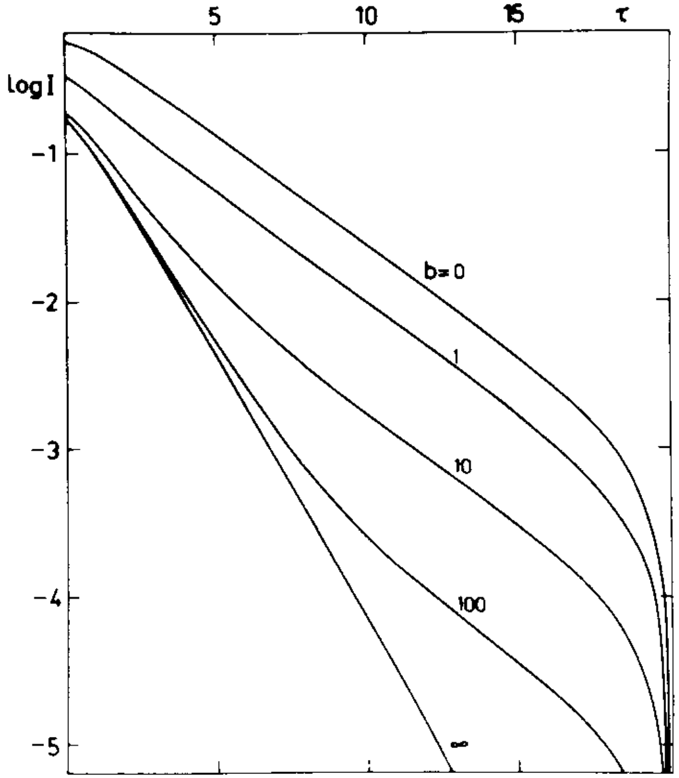

Fig. 1. The logarithm of the upward intensity versus optical depth for the standard problem in a finite atmosphere. The optical depth $\tau_0 = 20$ and arc cos μ = cos $\mu_0 \approx 12^{\circ}\!.9$. In Figures 1 to 4 the albedo of single scattering is given by formula (20).

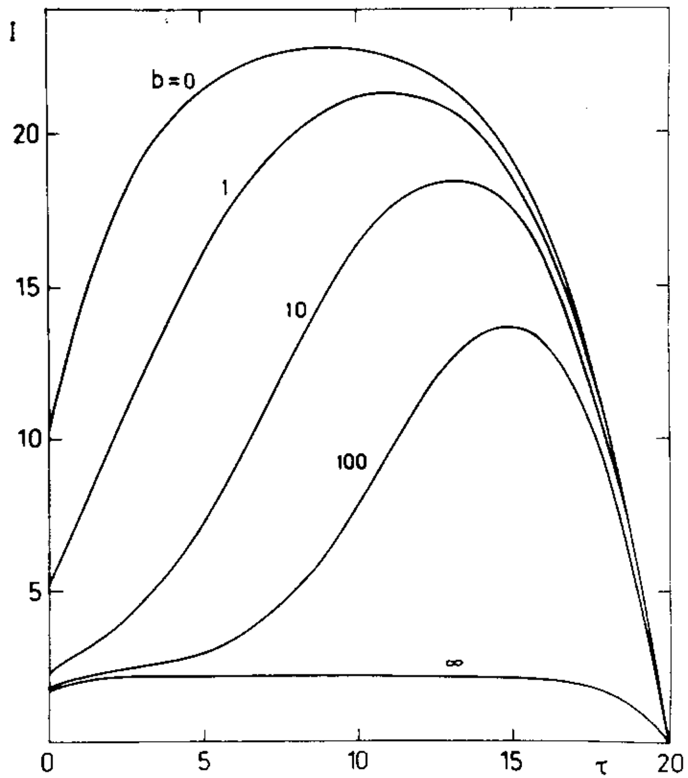

Fig. 2. The upward intensity versus optical depth in a finite atmosphere with internal sources. The internal source function is given by formula (16). $\tau_0 = 20$ and arc cos $\mu \approx 12^{\circ}\!.9$.

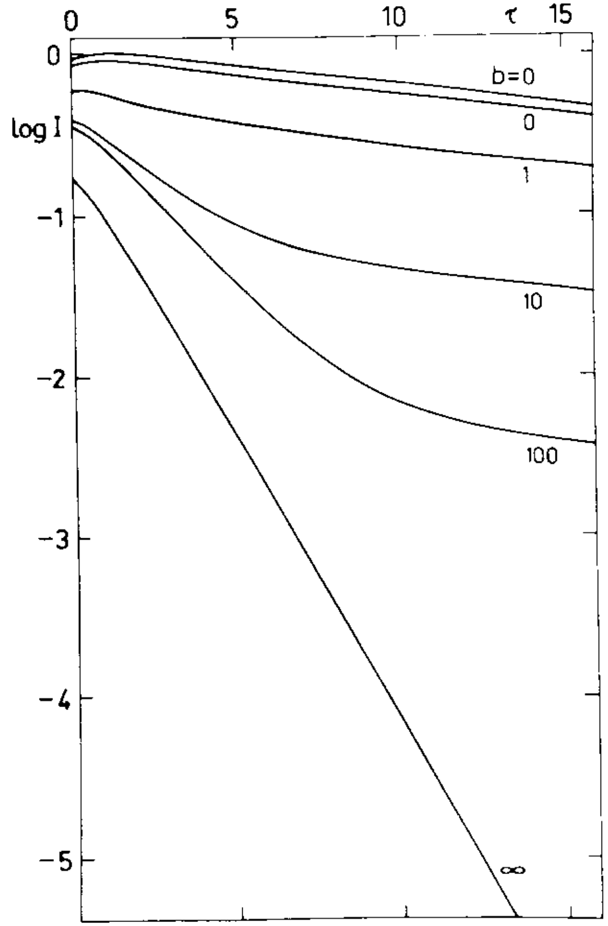

Fig. 3. The logarithm of the upward intensity versus optical depth for the standard problem in a semi-infinite atmosphere (arc cos μ = arc cos μ_0 = 12°9).

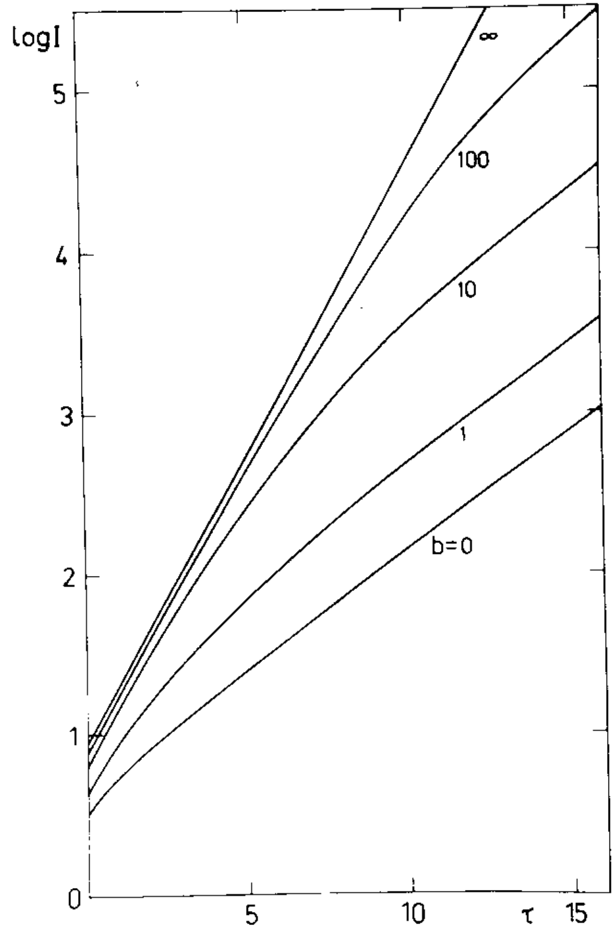

Fig. 4. The logarithm of the upward intensity versus optical depth for the Milne problem (arc cos $\mu \approx$ 12°9).

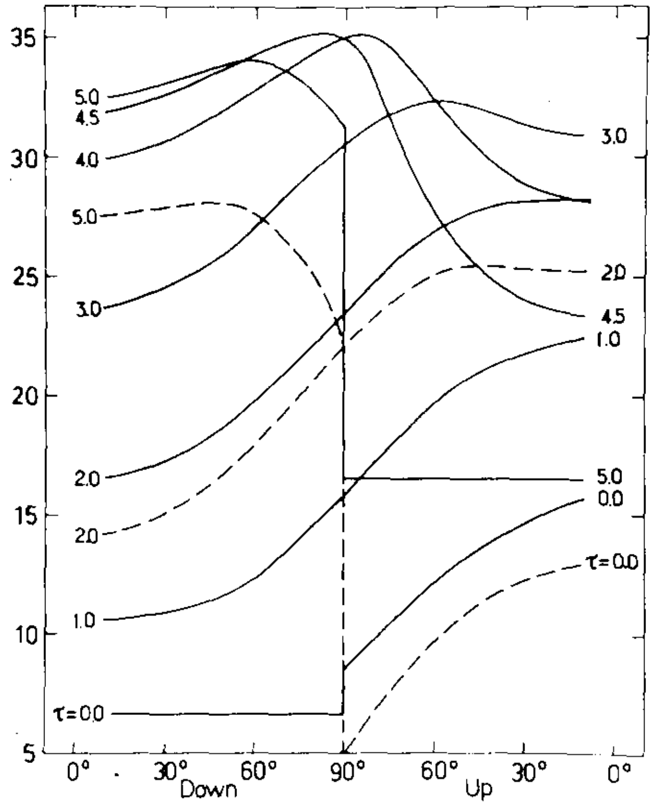

Fig. 5. The intensities in a finite homogeneous atmosphere bounded by reflecting Lambert boundaries. For the upper boundary $A = 1.0$, for the lower boundary $A = 0.5$. The albedo of single scattering is 0.8 and the internal source function is given by the formula $B_0(\tau) = 1 + 2\tau$. The dashed lines represent the intensities in the same atmosphere without the reflecting boundaries.

7. Conclusions

A generalized principle of invariance has been derived for a plane-parallel atmosphere. On the basis of this principle it has been possible to elaborate a method for determining the radiation field in a multi-layer atmosphere. This method can be used for optically finite and semi-infinite atmospheres. If we have selected the gaussian quadrature formula of order N then we have to make an inversion of an $N \times N$ matrix and to store an $N \times N$ matrix for every layer.

This method is most suitable for those atmospheres which consist of more or less homogeneous parts. Since the generalized principle of invariance is valid for the case of anisotropic scattering the method is applicable to more realistic cases with anisotropic scattering as well.

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