

## New Methods for Determining the Radiation Field in a Slab

T. Viik

Two new methods are proposed for determining the radiation field in a homogeneous slab with exponentially distributed sources. The approach is based on Chandrasekhar's principles of invariance.

### 1. Introduction

In this paper we present two new methods for determining the internal radiation field in a homogeneous slab with exponentially distributed sources. The techniques to be described are based on Chandrasekhar's principles of invariance [1, § 50]. The first approach has been used by Ivanov [2, 3] for a semi-infinite medium.

First we consider the homogeneous slab which is illuminated by uniform parallel rays of net flux  $\pi F$ . The slab absorbs radiation and scatters it isotropically.

In this case the mathematical expressions of Chandrasekhar's principles of invariance for the intensities in the outward and inward directions at any level  $\tau$  are

$$I(\tau, -v, u, x) = \frac{F}{4v} e^{-vu} S(x - \tau, v, u) + \frac{1}{4v} \int_0^1 S(x - \tau, v, \omega) I(\tau, \omega, u, x) d\omega, \quad (1)$$

$$I(\tau, v, u, x) = \frac{F}{4v} T(\tau, v, u) + \frac{1}{2v} \int_0^1 S(\tau, v, \omega) I(\tau, -\omega, u, x) d\omega, \quad (2)$$

where the angle of incidence arc  $\cos u$  and the angle of reflection arc  $\cos v$  are both measured with respect to the inward normal and  $S$  and  $T$  are the scattering and transmission functions, respectively.

Formula (1) means that the outward intensity at any level  $\tau$  results from the reflection of the reduced incident flux  $\pi F e^{-\tau u}$  and the diffuse radiation  $I(\tau, v, u, x)$  incident on the surface  $\tau$ , by the separate layer of optical thickness  $x - \tau$ , below  $\tau$ . Respectively, formula (2) means the inward intensity at any level  $\tau$  results from the transmission of the incident flux by the separate layer of optical thickness  $\tau$ , above the surface  $\tau$ , and the reflection by this same surface of the diffuse radiation  $I(\tau, -v, u, x)$  incident on it from below.

## 2. Formulation of the Cauchy Problem

Let us integrate formulae (1) and (2) with respect to  $v$ . Taking into account the expressions for the  $X$ - and  $Y$ -functions [1]

$$X(v, x) = 1 + \frac{1}{2} \int_0^1 S(x, v, \omega) d\omega / \omega, \quad (3)$$

$$Y(v, x) = e^{-xv} + \frac{1}{2} \int_0^1 T(x, v, \omega) d\omega / \omega, \quad (4)$$

and for the source function

$$B(\tau, u, x) = \frac{\lambda F}{4} e^{-\tau u} + \frac{\lambda}{2} \int_{-1}^{+1} I(\tau, \omega, u, x) d\omega, \quad (5)$$

we obtain

$$B(\tau, u, x) = \frac{\lambda F}{4} e^{-\tau u} X(u, x - \tau) + \frac{\lambda}{2} \int_0^1 X(\omega, x - \tau) I(\tau, \omega, u, x) d\omega \quad (6)$$

and

$$B(\tau, u, x) = \frac{\lambda F}{4} Y(u, \tau) + \frac{\lambda}{2} \int_0^1 X(\omega, \tau) I(\tau, -\omega, u, x) d\omega. \quad (7)$$

In equations (2) and (4) the  $X$ - and  $Y$ -functions are the well-known Ambartsumian-Chandrasekhar functions which satisfy the non-linear integro-differential equations [1]

$$\frac{\partial X(v, x)}{\partial x} = \frac{\lambda}{2} Y(v, x) \int_0^1 Y(\omega, x) d\omega/\omega, \quad (8)$$

$$\frac{\partial Y(v, x)}{\partial x} = -Y(v, x)/v + \frac{\lambda}{2} X(v, x) \int_0^1 Y(\omega, x) d\omega/\omega, \quad (9)$$

with initial conditions

$$X(v, 0) = 1, \quad Y(v, 0) = 1. \quad (10)$$

Considering the equation of transfer

$$v \frac{dI(\tau, v, u, x)}{d\tau} = -I(\tau, v, u, x) + B(\tau, u, x), \quad -1 \leq v \leq 1, \quad (11)$$

along with initial conditions

$$\begin{aligned} I(0, v, u, x) &= 0, \\ I(x, -v, u, x) &= 0, \end{aligned} \quad (12)$$

we see that owing to equations (6) and (7) we have reduced the two-point boundary value problem to an equivalent Cauchy problem. We have only to know the  $X$  and  $Y$ -functions to start the integration either from  $\tau=0$  when using formula (6) or from  $\tau=x$  when using formula (7). To determine the  $X$ - and  $Y$ -functions for the whole range of integration seems to be a small price to pay for this substantial simplification, especially when recalling the accurate numerical procedure of Bellman et al. [4] to compute these basic functions of radiative transfer. There is another simple way to calculate the  $X$  and  $Y$ -functions, using the approximate formulae [5] (or [6])

$$X(u, \tau) = p(u, \tau) + q(-u, \tau) e^{-\tau u}, \quad (13)$$

$$Y(u, \tau) = q(u, \tau) + p(-u, \tau) e^{-\tau u}, \quad (14)$$

where

$$p(u, \tau) = 1 + \sum_{i=1}^N \left[ \frac{a_i u}{1 + \beta_i u} + \frac{b_i u^{1+\delta(i-\lambda)} e^{-\beta_i \tau}}{1 - \beta_i u} \right], \quad (15)$$

$$q(u, \tau) = \sum_{i=1}^N \left[ \frac{a_i u e^{-\beta_i \tau}}{1 - \beta_i u} + \frac{b_i u (\tau - u)^{\delta(i-\lambda)}}{1 + \beta_i u} \right]. \quad (16)$$

Here

$$\delta(i-\lambda) = \begin{cases} 1, & \text{if } i=\lambda=1, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\beta_i$  are the solutions of the characteristic equation [7]

$$1 - \lambda \sum_{h=1}^N \frac{\omega_h}{1 - \beta^2 u^2} = 0. \quad (17)$$

In equation (17)  $\omega_h$  and  $u_h$  are the weights and the points of the Gaussian quadrature of order  $N$ , normalized to the interval  $(0, 1)$ . The coefficients  $a_i$  and  $b_i$  are to be determined from the system

$$\begin{aligned} p(-u_h) &= 0, \\ q(-u_h) &= 0, \quad k=1, 2, \dots, N. \end{aligned} \quad (18)$$

The approach given in [5] is powerful enough to determine the whole radiation field in a slab but here we give this reference because in the following we need the  $X$ - and  $Y$ -functions for the argument  $1 \leq u \leq \infty$ .

### 3. The Source Function

If we are interested only in the source function then by using the formal solutions for the outward and inward intensities

$$I(\tau, -v, u, x) = \int_{\tau}^x B(t, u, x) e^{-\frac{t-\tau}{v}} dt/v, \quad (19)$$

$$I(\tau, v, u, x) = \int_0^{\tau} B(t, u, x) e^{-\frac{\tau-t}{v}} dt/v, \quad (20)$$

and formulae (6) and (7) we may derive Volterra integral equations of the second kind for the source function in the form

$$B(\tau, u, x) = \frac{\lambda F}{4} e^{-\tau/u} X(u, x - \tau) +$$

$$+\frac{\tau}{2} \int_0^{\tau} W(\tau-t, x-\tau) B(t, u, x) dt \quad (21)$$

and

$$B(\tau, u, x) = \frac{\lambda F}{4} Y(u, \tau) + \frac{\lambda}{2} \int_{\tau}^x Q(t-\tau, \tau) B(t, u, x) dt, \quad (22)$$

where

$$W(\tau, t) = \int_0^1 X(\omega, t) e^{-\tau/\omega} d\omega/\omega \quad (23)$$

and

$$Q(t, \tau) = \int_0^1 X(\omega, \tau) e^{-t/\omega} d\omega/\omega. \quad (24)$$

Unlike the well-known Fredholm integral equation for the source function

$$B(\tau, u, x) = \frac{\lambda F}{4} e^{-\tau/u} + \frac{\lambda}{2} \int_0^x E_1(|t-\tau|) B(t, u, x) dx, \quad (25)$$

where

$$E_1(s) = \int_0^1 e^{-\omega/s} d\omega/\omega, \quad (26)$$

which express the source function at the level  $\tau$  through the values of the source function at all levels, equations (21) and (22) express the source function at the level  $\tau$  only through its values at lower or higher levels than  $\tau$ .

It is interesting to point out that Danielian [9] has derived equation (21) using the generalized Ambartsumian principle of invariance [10].

#### 4. The Solution of the Cauchy Problem

To solve the above Cauchy problem, we integrated numerically the equation of transfer for the downward intensity

$$v \frac{dI(\tau, v, u, x)}{d\tau} = -I(\tau, v, u, x) + \quad (27)$$

$$+\frac{\lambda}{2} \int_0^1 X(\omega, x-\tau) I(\tau, \omega, u, x) d\omega + \frac{\lambda F}{4} e^{-\tau/u} X(u, x-\tau), \quad v \geq 0,$$

along with the boundary condition

$$I(0, v, u, x) = 0 \quad (28)$$

and the system (8)–(9) or (13)–(14).

The upward intensity has been obtained in two ways. First, Chandrasekhar's principle of invariance (1) has been used taking into account that

$$S(x, v, u) = \frac{\lambda uv}{u+v} [X(u, x)X(v, x) - Y(u, x)Y(v, x)]. \quad (29)$$

Second, equation (7) has been used in the equation of transfer for the upward intensity

$$v \frac{dI(\tau, -v, u, x)}{d\tau} = I(\tau, -v, u, x) - \frac{\lambda}{2} \int_0^1 X(\omega, \tau) I(\tau, -\omega, u, x) d\omega - \frac{\lambda F}{4} Y(u, \tau), \quad (30)$$

with the boundary condition

$$I(x, -v, u, x) = 0.$$

The integrals in the right-hand side of equations (27) and (30), as well as the integrals in equations (8) and (9) were approximated by finite sums through Gaussian quadrature of the order  $N$ . The four-point Runge-Kutta method was used with the stepsize of 0.005 and  $N=7$  in our calculations.

Having determined the downward and upward intensities, the source function has been obtained from equation (7). The accuracy of computation has been checked against the independent results for the source function determined in a different way [5]. The results of comparison for different optical depths, angles of incidence and albedos of single scattering showed that for the slab thickness  $x=1$  and the step size  $\Delta\tau=0.005$  the relative errors of the source function were never worse than 0.4%, and generally much better.

The determination of the upward intensity using the Chandrasekhar's principle turned out to be slightly more accurate than using equation (30).

### 5. The Problem with Exponentially Distributed Internal Sources

The above-mentioned approach is powerful enough to cope with the problem of determining the radiation field in a slab with exponentially distributed internal sources. In this case

$$B(\tau, x) = (1 - \lambda) B e^{-m\tau} + \frac{\lambda}{2} \int_{-1}^{+1} I(\tau, \omega, x) d\omega, \quad (31)$$

where  $B = \text{const.}$

If we put

$$u = m^{-1}$$

and

$$F = 4\lambda^{-1}(1 - \lambda)B,$$

we may use the technique given above. To find the respective values of the  $X$ - and  $Y$ -functions in case of  $0 \leq |m| < 1$  we are forced to use the method described in [5] since equations (8) and (9) are of no use any more.

From equations (5) and (6) we find the source function in the form

$$B(\tau, x) = (1 - \lambda) B e^{-m\tau} X(m^{-1}, x - \tau) + \frac{\lambda}{2} \int_0^1 X(\omega, x - \tau) I(\tau, \omega, x) d\omega, \quad (32)$$

$$B(\tau, x) = (1 - \lambda) B Y(m^{-1}, \tau) + \frac{\lambda}{2} \int_0^1 X(\omega, \tau) I(\tau, -\omega, x) d\omega. \quad (33)$$

For  $m < 0$  the calculation of the  $X$ - and  $Y$ -functions for negative arguments can be avoided by using the formulae [13]

$$\begin{aligned} X(-u, \tau) &= e^{i u} Y(u, \tau) \\ Y(-u, \tau) &= e^{i u} X(u, \tau). \end{aligned}$$

The important case of uniformly distributed internal sources is characterized by equations (31), (32) and (33) with  $m = 0$ . In this case the values of the source function at the boundaries are equal,

$$B(0, x) = B(x, x) = (1 - \lambda) B X(\infty, x), \quad (34)$$

where the  $X$ - and  $Y$ -functions at  $u \rightarrow \infty$  are easily obtained from equations (13)–(16),

$$X(\infty, t) = Y(\infty, t) = 1 + \sum_{i=1}^N \beta_i^{-1} (a_i + b_i) (1 - e^{-\beta_i t}), \quad \lambda \neq 1. \quad (35)$$

Again, solving equations (27) and (30) along with the vacuum boundary conditions we may determine the whole radiation field in a slab with exponentially or uniformly distributed internal sources.

We checked the accuracy of the source function at  $\tau=0$  against the results obtained by using the formula for the emergent intensity in a slab with exponentially distributed internal sources [8]

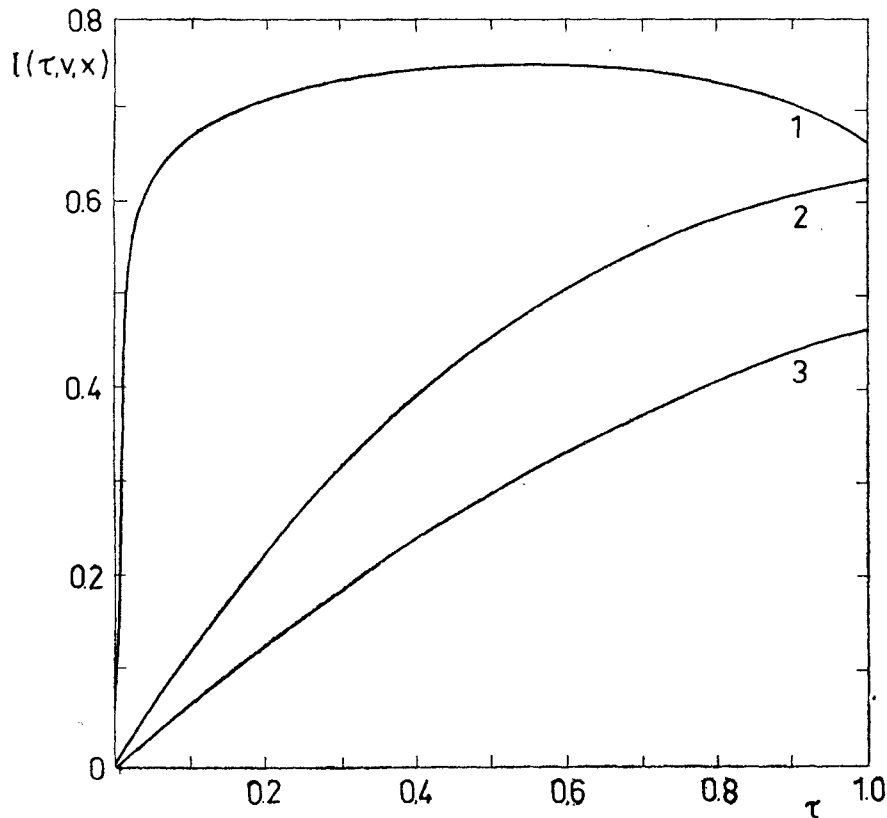


Fig. 1. The downward intensity at various optical depths in an atmosphere with uniformly distributed sources: 1 —  $\nu = 12^\circ.9$ , 2 —  $\nu = 60^\circ$ , 3 —  $\nu = 88^\circ.5$ .

$$I(0, -v, x) = 4\lambda^{-1}v^{-1}(1 - \lambda)B \int_0^x e^{-mt} B(t, v, x) dt, \quad (36)$$

where the source function  $B(\tau, v, x)$  has been obtained by the method described in [5]. Unfortunately we had to confine ourselves to the checking of accuracy of the source function only at  $\tau=0$  since there is no other simple method for determining the radiation field accurately enough in a slab with exponentially distributed internal sources.

For  $\lambda=0.5$ ,  $x=1$ ,  $B=1$ ,  $\Delta\tau=0.005$  and  $m=0.9$  the relative error of the source function at  $\tau=0$  is 0.09%, but for the same parameters with  $m=-0.9$  the relative error is 2.4%.

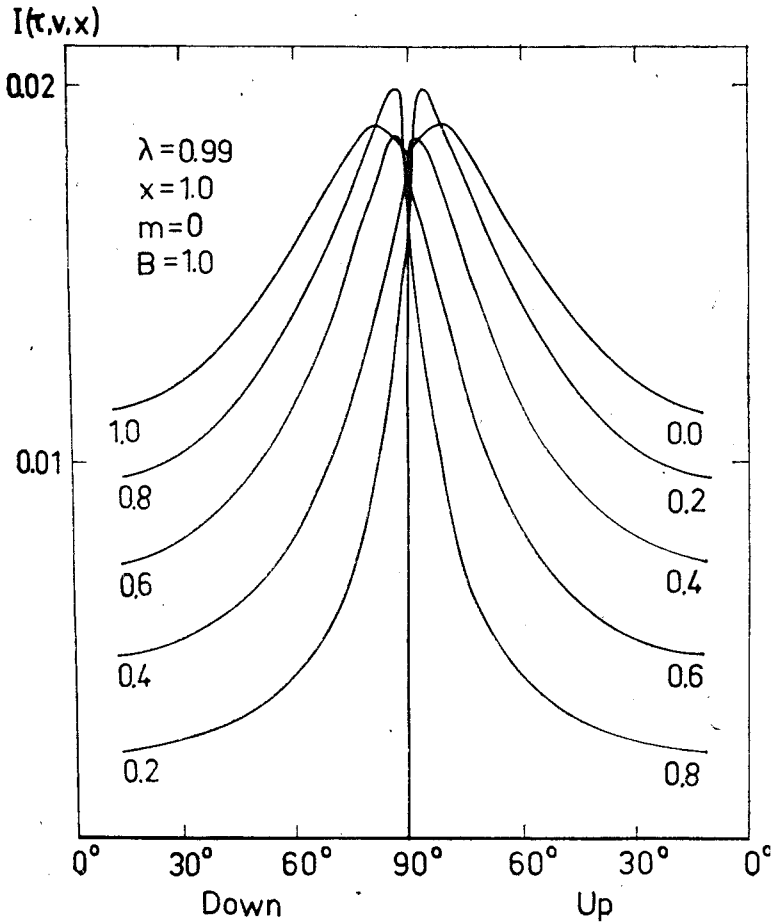


Fig. 2. The intensity versus angle of reflection at various optical depths for  $\lambda=0.99$ ,  $x=1.0$ ,  $m=0$  and  $B=1$ .

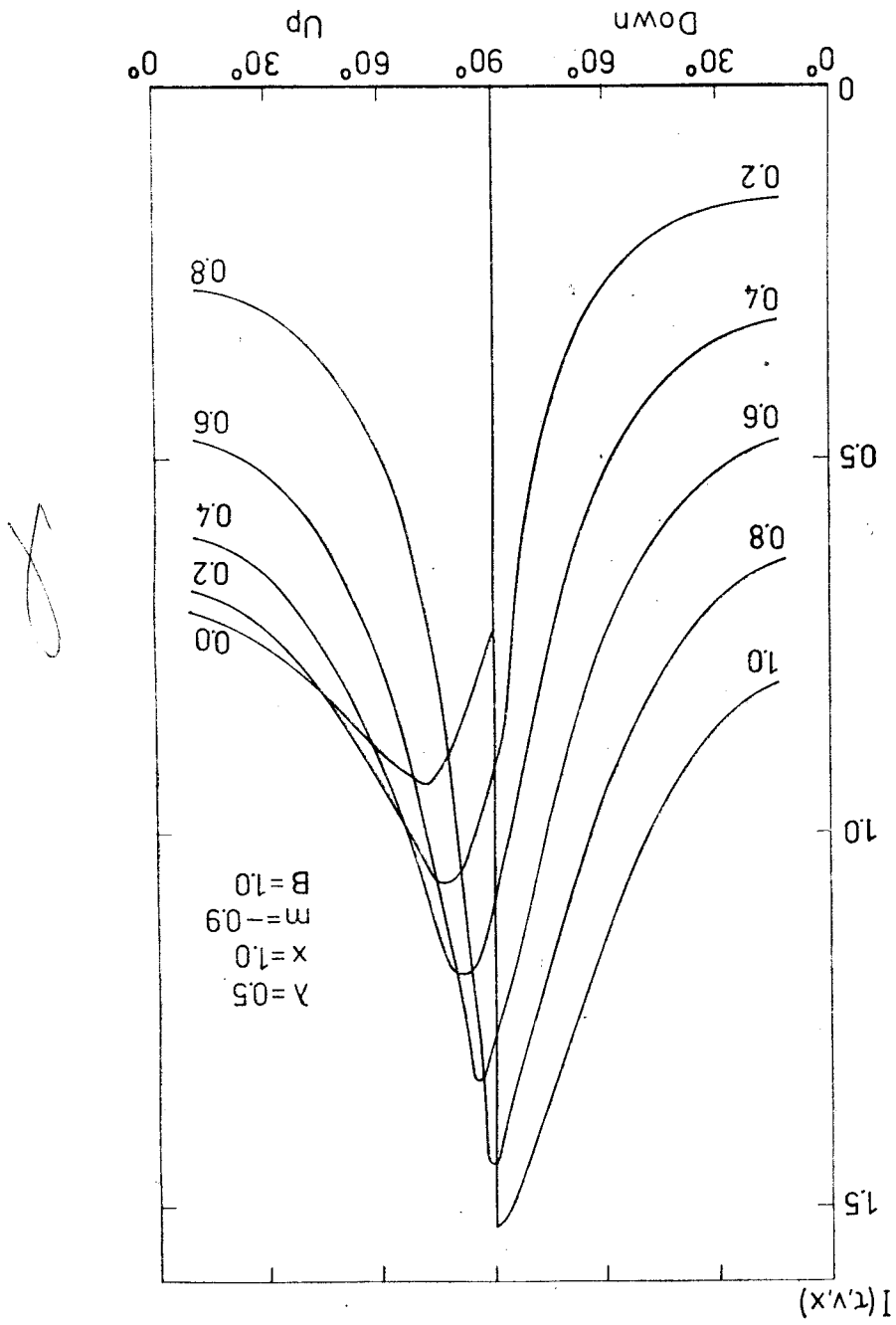


Fig. 3. The intensity versus angle of reflection at various optical depths for  $\lambda=0.5$ ,  $x=1.0$ ,  $m=-0.9$  and  $B=1$

In the case of uniformly distributed sources with  $\lambda=0.99$ ,  $x=1$ ,  $B=1$ ,  $\Delta\tau=0.005$  the relative error of the source function at  $\tau=0$  is 5.7%. For the same parameters with  $\lambda=0.5$  the relative error at  $\tau=0$  is as small as 0.09% and for  $\lambda=0.01$  —  $10^{-6}$ %. For three angles  $v_1=88^\circ.5$ ,  $v_2=60^\circ$  and  $v_3=12^\circ.9$  the run of the downward intensity versus optical depth is given in Fig. 1. Note the internal maximum of the first curve approximately at the center of the slab. Figs. 2 and 3 show internal intensity variations at different optical depths for  $m=0$  and  $m=-0.9$ .

## 6. The Problem with Radiating Surfaces

Having solved the standard problem we are able to obtain solutions for some other problems. For instance, let us consider an absorbing and scattering homogeneous slab without internal sources. The radiation field in the slab is due to the radiating boundary at  $\tau=x$ ,

$$I(x, -v, x) = \varrho(v). \quad (37)$$

In this case the source function in terms of the diffuse intensity is

$$B(\tau, x) = \frac{\lambda}{2} \int_{-1}^{+1} I(\tau, v, x) dv + B^*(\tau, x), \quad (38)$$

where

$$B^*(\tau, x) = \frac{\lambda}{2} \int_0^1 \varrho(v) e^{-\frac{x-\tau}{v}} dv. \quad (39)$$

Since we may present the boundary condition at  $\tau=x$  (equation (37)) as a superposition of delta-function beams

$$I(x, -v, x) = \int_0^1 \varrho(u) \delta(v-u) du, \quad (40)$$

the diffuse intensity at any point in the slab is also a superposition of the intensities of the standard problem,

$$I(\tau, v, x) = \frac{2}{F} \int_0^1 \varrho(u) I(x-\tau, -v, u, x) du, \quad -1 \leq v \leq 1. \quad (41)$$

### 7. The Functions $b$ and $h$ .

An important special case of radiating boundaries

$$\varrho(v) = v^{-1} \quad (42)$$

both at  $\tau=0$  and  $\tau=x$  has been considered by Kagiwada and Kalaba [11]. They introduced the functions  $b$  and  $h$  which in our notation are as follows

$$b(\tau, v, x) = \frac{1}{v} e^{-\tau/v} \vartheta(v) + \frac{2}{F} \int_0^1 I(\tau, v, u, x) du/u, \quad (43)$$

$$h(\tau, v, x) = \frac{1}{|v|} e^{-\frac{x-\tau}{|v|}} \vartheta(-v) + \frac{2}{F} \int_0^1 I(x-\tau, -v, u, x) du/u, \quad (44)$$

where  $-1 \leq v \leq 1$  and  $\vartheta(v)$  is the step function

$$\vartheta(v) = \begin{cases} 1, & v \geq 0, \\ 0, & v < 0. \end{cases}$$

Actually there is only one function, either  $b$  or  $h$ , since

$$\begin{aligned} b(\tau, v, x) &= h(x-\tau, -v, x) \\ h(\tau, v, x) &= b(x-\tau, -v, x), \end{aligned} \quad (45)$$

as we may easily deduce from equations (43) and (44). Therefore we shall use only the  $b$ -function. Kagiwada and Kalaba have shown that the solution of isotropic source problem, i. e. obtaining the  $b$ -function gives directly the solution of the standard problem [14]. The importance of the  $b$ -function is accentuated by the fact that it is a vector, compared with the intensity  $I(\tau, v, u, x)$  which is a matrix at every optical depth. In order to determine the whole radiation field in a slab using the method described in the previous parts of this paper we have to solve the above equations for every angle of incidence separately. Using the  $b$ -function we have to do it only once.

Kagiwada and Kalaba have formulated the Cauchy problem for the  $b$ -function using the invariant imbedding methods [11]. In the following we shall describe another approach to find the  $b$ -function utilizing Chandrasekhar's principles of invariance.

### 8. Chandrasekhar's Principles for the $b$ -Function

Using Chandrasekhar's principles of invariance (1) and (2) for the standard problem, we find the respective principles for the problem with isotropic sources

$$b(\tau, -v, x) = \frac{1}{2v} \int_0^1 S(x - \tau, v, \omega) b(\tau, \omega, x) d\omega, \quad (46)$$

$$b(\tau, v, x) = \frac{1}{v} Y(v, \tau) + \frac{1}{2v} \int_0^1 S(\tau, v, \omega) b(\tau, -\omega, x) d\omega. \quad (47)$$

Here we have taken into account equation (4) and the definition of  $b$ -function (43). Chandrasekhar's principles for the  $b$ -function take especially simple form and they throw more light on the nature of the reflection function  $S$ .

### 9. The Equation of Transfer for the $b$ -Function

Kagiwada and Kalaba have derived the equation of transfer for the  $b$ -function [11]

$$v \frac{db(\tau, v, x)}{d\tau} = -b(\tau, v, x) + \Phi(\tau, x), \quad (48)$$

where the source function  $\Phi$  is the resolvent function of Sobolev [8] and it is given by

$$\Phi(\tau, x) = \frac{2}{F} \int_0^1 B(\tau, u, x) du/u. \quad (49)$$

Using the definition for the source function  $B$  (equation (5)) and equation (43) the resolvent function is rewritten in the form

$$\Phi(\tau, x) = \frac{\lambda}{2} \int_{-1}^{+1} b(\tau, v, x) dv. \quad (50)$$

In the following we need also the boundary values of the  $b$ -function. To obtain them we use equations (3), (4), (12) and the definitions of the scattering and transmission function [1]

$$I(0, -v, u, x) = \frac{F}{4v} S(x, v, u), \quad (51)$$

$$I(x, v, u, x) = \frac{F}{4v} T(x, v, u).$$

The result is

$$\begin{aligned} vb(0, v, x) &= 1, \\ vb(0, -v, x) &= X(v, x) - 1, \\ vb(x, v, x) &= Y(v, x), \\ vb(x, -v, x) &= 0. \end{aligned} \quad (52)$$

We see that the boundary values of the  $b$ -function are uniquely defined at both boundaries.

### 10. Formulation of the Cauchy Problem for the $b$ -Function

Now we are able to use the same way as we have already done when obtaining equations (3)–(7) to separate the  $b$ -function for radiation travelling in upward and downward directions in equation (48).

Integrating Chandrasekhar's principles of invariance for the  $b$ -function (46) and (47) over  $v$  we obtain

$$\Phi(\tau, x) = \frac{\lambda}{2} \int_0^1 X(\omega, x - \tau) b(\tau, \omega, x) d\omega \quad (53)$$

and

$$\Phi(\tau, x) = \frac{\lambda}{2} \int_0^{\frac{x}{\tau}} X(\omega, \tau) b(\tau, -\omega, x) d\omega + \frac{\lambda}{2} \int_0^1 Y(\omega, \tau) d\omega/\omega. \quad (54)$$

Equations (48), (53) and (54) along with the boundary conditions (52) define the Cauchy problem for the  $b$ -function. After having found the  $b$ -function both in upward and downward directions, the Sobolev  $\Phi$ -function can be obtained by means of equation (52). According to equation (48) this gives us the  $b$ -function in the horizontal direction. At the top of the slab both the  $\Phi$ - and  $b$ -functions in the horizontal directions are infinite.

Kagiwada and Kalaba have shown that the functions  $X$ ,  $Y$ ,  $S$ ,  $T$ ,  $\Phi$ ,  $B$  and  $I$  are all expressible algebraically in terms of the  $b$ -function [14].

Since we have to determine the  $X$ - and  $Y$ -functions anyway, we give here only the formulas for  $\Phi$ ,  $B$  and  $I$ :

Päikese raadius $R_{\odot}$	
Päikese efektiivne temp	
Päikese absoluutne bo	
Päikese näiv bolomeet	
Päikese näiv ultraviole	
Päikese näiv sinine täh	
Päikese näiv visuaalne	
Maa mass $M_{\oplus}$	$= 5.972$
Maa raadius $R_{\oplus}$	$= 6.3$
Valgusaasta $ly$	$= 9.460$
Parsek $pc$	$= 206264.806$
Astronoomiline ühik $AU$	
Sideeriline päev	$= 23^h$
Päikeseööpäev	$= 86400$
Sideeriline aasta	$= 3.155$
Troopiline aasta	$= 3.155$
Julise aasta	$= 3.155760$
Gregorjuse aasta	$= 3.155$

$$\Phi(\tau, x) = b(\tau, 0, x), \quad (55)$$

$$B(\tau, u, x) = \frac{\lambda Fu}{4} [X(u, x)b(\tau, u, x) - Y(u, x)b(x - \tau, -u, x)], \quad (56)$$

$$\left(\frac{1}{u} - \frac{1}{v}\right) I(\tau, v, u, x) = -\frac{1}{v} B(\tau, u, x) + \quad (57)$$

$$+\frac{\lambda F}{4} [X(u, x)b(\tau, v, x) - Y(u, x)b(x - \tau, -v, x)].$$

## 11. The Numerical Procedure

To solve equation (48) with the source term (53) we approximate the integral in equation (53) by a finite sum of order  $N$ . If we choose  $N=7$ , as Bellman did in similar calculations [4], equation (48) for  $v \geq 0$  becomes a system of seven ordinary differential equations subject to the initial condition

$$v_i b(0, v_i, x) = 1, \quad i=1, 2, \dots, 7. \quad (58)$$

We solved this system by the fourth-order Runge-Kutta procedure starting from  $\tau=0$  with the step size  $\Delta\tau=0.005$ . The results appeared to be in a good agreement with those of Kagiwada and Kalaba [11]. To find the  $b$ -function in upward directions we tried to integrate equation (48) along with the source term (54) for  $v < 0$  starting from  $\tau=0$  and using as a boundary condition

$$v_i b(0, -v_i, x) = X(v_i, x) - 1, \quad i=1, 2, \dots, 7. \quad (59)$$

Unfortunately the system under these conditions appeared to be completely unstable. Integrating of (48) from  $\tau=x$  and using boundary conditions

$$b(x, -v_i, x) = 0, \quad i=1, 2, \dots, 7, \quad (60)$$

gave good results.

Still better results for the  $b$ -function in upward directions were obtained when using Chandrasekhar's principle (46) and the expression for the  $S$ -function (29).

We have found also Sobolev's  $\Phi$ -function according to formula (50) which seems to be an easier way than extrapolating the  $b$ -function according to formula (55).

## 12. Conclusion

In this paper we have presented two new approaches for determining the radiation field in a homogeneous slab with exponentially distributed sources. Both approaches are based on Chandrasekhar's principles of invariance. Several numerical computations have been made and the accuracy of proposed methods is found to be sufficiently high. If one is interested in the accuracy still higher then it may be achieved by decreasing the step size of numerical integration. The problem shows some features of the "stiff" systems, i. e. the orders of magnitude of the eigenvalues of the system are very different. This difficulty may become serious when dealing with optically very thick slabs.

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## Новые методы определения поля излучения в слое

Т. Вийк

Предлагаются два новых метода для определения поля излучения в однородном слое с экспоненциально распределенными источниками. Методы базируются на использовании принципов инвариантности Чандрасекара.

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