ПУБЛИКАЦИИ ТАРТУСКОЙ АСТРОФИЗИЧЕСКОЙ ОБСЕРВАТОРИИ им. В. СТРУВЕ

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The Milne and the Albedo Problems in the P-5 Approximation

T. Viik

The Milne and the albedo problems are solved in the P-5 approximation, using different boundary conditions. It is shown that the Pomraning boundary conditions are to be preferred when considering the radiation field far from boundaries and if one is not dealing with the case of grazing incidence.

1. First we consider the Milne problem where a pure isotropic scatterer occupies the half-space with a source at infinity, in the P-5 approximation using different boundary conditions.

The equation of radiative transfer is

$$\mu \frac{dI}{d\tau} + I(\tau, \mu) = \frac{1}{2} \int_{-1}^{+1} I(\tau, y)\, dy, \tag{1}$$

where the angle variable μ is measured with respect to the positive τ-axis. The other notations have their customary meanings.

The moment equations in the P-N approximation are the following [1]:

$$\begin{aligned} &\psi'_1(\tau) = 0, \\ &n\psi'_{n-1}(\tau) + (2n+1)\psi_n(\tau) + (n+1)\psi'_{n+1}(\tau) = 0, \qquad n \geq 1, \end{aligned} \tag{2}$$

where $\psi_n(\tau)$ is defined by the series expansion

$$I(\tau, \mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\mu)\psi_n(\tau).$$

In the P-5 approximation system (2) can be reduced to a sixth-order differential equation with constant coefficients.

$$75\psi_0^{\mathrm{VI}}(\tau) - 882\psi_0^{\mathrm{IV}}(\tau) + 1155\psi_0''(\tau) = 0. \tag{3}$$

The general solution of equation (3) which has the proper behaviour at infinity is

$$\psi_0(\tau) = A_1 + A_2\tau + A_3\, e^{-\lambda_1\tau} + A_4\, e^{-\lambda_2\tau}. \tag{4}$$

Here the coefficients A_i are the constants of integration and λ_i are the positive roots of the characteristic equation

$$75\lambda^4 - 882\lambda^2 + 1155 = 0, \tag{5}$$

so that

$$\lambda_1 = 3.2029452, \qquad 3.202945254 \qquad +0.000000071$$
$$\lambda_2 = 1.2252108. \qquad 1.2252108\,8 \qquad 0.000000552$$

Then the other ψ_i are:

$$\psi_1(\tau) = -\frac{1}{3} A_2,$$

$$\psi_2(\tau) = -\frac{1}{2}(A_3\, e^{-\lambda_1\tau} + A_4\, e^{-\lambda_2\tau}),$$

$$\psi_3(\tau) = -\frac{5}{6}(\lambda_1^{-1} A_3\, e^{-\lambda_1\tau} + \lambda_2^{-1} A_4\, e^{-\lambda_2\tau}),$$

$$\psi_4(\tau) = p_1 A_3\, e^{-\lambda_1\tau} + p_2 A_4\, e^{-\lambda_2\tau},$$
$$\psi_5(\tau) = q_1 A_3\, e^{-\lambda_1\tau} + q_2 A_4\, e^{-\lambda_2\tau},$$

$p_1 = 0.2323764298$
$p_2 = -0.596482794$
$q_1 = 0.3381174316$
$q_2 = -0.332183 6411$

where

$$p_i = \frac{1}{8}\left(3 - \frac{35}{3}\lambda_i^2\right)$$

and

$$q_i = \frac{1}{40}\left(\frac{161}{3} - 105\lambda_i^2\right)\lambda_i^{-1}, \qquad i = 1,\ 2.$$

In order to determine the coefficients A_i, we first derive the boundary conditions according to Pomraning [2]:

$$\int_0^1 I(0, y)\, \delta I(0, -y)\, y\, dy = 0, \tag{6}$$

where δI is the variation of the specific intensity. Since the specific intensity in the P-5 approximation is

$$2I(\tau, \mu) = \psi_0(\tau) + 3P_1(\mu)\psi_1(\tau) + 5P_2(\mu)\psi_2(\tau) +$$
$$+ 7P_3(\mu)\psi_3(\tau) + 9P_4(\mu)\psi_4(\tau) + 11P_5(\mu)\psi_5(\tau),$$

equation (6) yields

$$
\begin{aligned}
&128\langle 00\rangle-256\langle 01\rangle+160\langle 02\rangle-48\langle 04\rangle+256\langle 10\rangle-576\langle 11\rangle+\\
&+512\langle 12\rangle-224\langle 13\rangle+44\langle 15\rangle+160\langle 20\rangle-512\langle 21\rangle+\\
&+800\langle 22\rangle-768\langle 23\rangle+390\langle 24\rangle-224\langle 31\rangle+768\langle 32\rangle-\\
&-1176\langle 33\rangle+1024\langle 34\rangle-462\langle 35\rangle-48\langle 40\rangle+390\langle 42\rangle-\\
&-1024\langle 43\rangle+1458\langle 44\rangle-1280\langle 45\rangle+\\
&+44\langle 51\rangle-462\langle 53\rangle+1280\langle 54\rangle-1815\langle 55\rangle=0,
\end{aligned}
$$

where

$$\langle ik\rangle=\psi_i(0)\,\delta\psi_k(0).$$

Not all the variations $\delta\psi_k$ are independent. The restrictions imposed on them are given by the following equations:

$$\psi_n(0)=\sum_{m=0}^{2}(-1)^n R_{6-n,m}\psi_m(0),\qquad n=5,\ 4,\ 3. \tag{7}$$

Due to the zero entrant flux at $\tau=0$ there are no inhomogeneous terms in equations (7). According to equations (7) there are only nine independent combinations of $\langle ik\rangle$.

Setting the coefficients of these combinations equal to zero yields nine nonlinear equations for the nine unknowns R_{ik} $(i,k=0,1,2)$:

$$
\begin{aligned}
&1815R_{10}^2-1458R_{20}^2+1176R_{30}^2+924R_{10}R_{30}+96R_{20}-128=0,\\
&1815R_{11}^2-1458R_{21}^2+1176R_{31}^2+924R_{11}R_{31}+88R_{11}-448R_{31}+576=0,\\
&1815R_{12}^2-1458R_{22}^2+1176R_{32}^2+924R_{12}R_{32}-780R_{22}-800=0,\\
&4(R_{20}R_{31}-R_{21}R_{30})+5(R_{11}R_{20}-R_{10}R_{21})-1=0,\\
&4(R_{21}R_{32}-R_{22}R_{31})+5(R_{12}R_{21}-R_{11}R_{22})-3R_{31}+2=0,\\
&4(R_{20}R_{32}-R_{22}R_{30})+5(R_{12}R_{20}-R_{10}R_{22})-3R_{30}=0,\\
&44R_{10}+48R_{21}-224R_{30}+1815R_{10}R_{11}-1458R_{20}R_{21}+1176R_{30}R_{31}+\\
&+462(R_{11}R_{30}+R_{10}R_{31})=0, \qquad\qquad (8)\\
&44R_{12}-390R_{21}-224R_{32}+1815R_{11}R_{12}-1458R_{20}R_{21}+1176R_{31}R_{32}+\\
&+462(R_{12}R_{31}+R_{11}R_{32})=0,\\
&390R_{20}+48R_{22}-160+1815R_{10}R_{12}-1458R_{20}R_{22}+1176R_{30}R_{32}+\\
&+462(R_{12}R_{30}+R_{10}R_{32})=0.
\end{aligned}
$$

It is quite possible that system (8) has an analytic solution as the respective system in the P-3 approximation has [3]. However, it is much easier to solve the system numerically. In that case the most difficult problem is the determination of the initial values of R_{ik} before starting iteration. Fortunately the number of unknowns to be used as an initial guess can be reduced to three, when taking advantage of the following procedure. The unknown R_{10} can be found as a function of R_{20} and R_{30} from the first equation of system (8). From the second equation we find $R_{21}(R_{11}, R_{31})$. Using the unknowns R_{10} and R_{21} in the fourth equation, we obtain a quadratic equation to determine R_{20}. The coefficients of this equation contain only three unknowns: R_{11}, R_{30} and R_{31}. Hence, starting with the estimates for these unknowns, we can successively find approximate values for other unknowns. Having found a suitable set of R_{ik}, we can start iteration. The best rate of convergence was obtained when using the Newton method.

The solution of system (8) along with the respective values for the coefficients R_{ik} in the Mark and Marshak approximations are given in Table 1.

Table 1

R_{ik}	MARK	MARSHAK	POMRANING
R_{10}	3.048290	1.750000	1.368250
R_{11}	5.809780	3.375000	2.665275
R_{12}	2.490743	1.914062	1.544624
R_{20}	3.746741	2.666667	2.307220
R_{21}	7.347313	5.333333	4.665505
R_{22}	4.384723	3.333333	2.986179
R_{30}	1.894722	1.562500	1.442332
R_{31}	3.992101	3.375000	3.151829
R_{32}	3.053831	2.734375	2.618466

We note that the Pomraning values of R_{ik} follow the same trend as the Mark and Marshak values do, but they are smaller in absolute value.

It is interesting to point out that the second triplet of system (8) is satisfied when using the Marshak values of R_{ik}.

Thus, the coefficients A_i can be determined from equations

Table 3

Fractional errors of the mean intensity expressed as precentages. The Milne Problem

τ	P-1			P-3			P-5		
	MARK	MARSHAK	POMRANING	MARK	MARSHAK	POMRAN-ING	MARK	MARSHAK	POMRAN-ING
0	0.00	—15.5	—22.5	0.00	—4.38	—7.06	—3.15	—2.51	—4.22
0.01	1.82	—13.1	—19.9	1.44	—2.74	—5.31	—1.64	—1.13	—2.75
0.05	5.06	—8.46	—14.6	3.40	—0.22	—2.45	0.52	0.61	—0.73
0.10	6.95	—5.32	—10.9	4.08	0.96	—0.96	1.43	1.14	0.06
0.20	8.50	—2.01	—6.77	4.02	1.59	0.10	1.61	1.09	0.33
0.30	8.93	—3.43	—4.54	3.52	1.56	0.35	1.70	0.80	0.24
0.50	8.72	1.15	—2.27	2.52	1.14	0.29	1.34	0.37	0.02
1.00	7.13	1.88	—0.50	1.22	0.45	—0.02	0.81	0.09	—0.08
2.00	4.82	1.52	0.03	0.60	0.18	—0.08	0.45	0.08	—0.03
3.00	3.57	1.16	0.07	0.43	0.13	—0.05	0.31	0.06	—0.02

For small optical depths the errors are rather large. In this case one may prefer the Marshak boundary conditions.

The statement that the Marshak boundary conditions are more accurate that those of the Mark in lower-order approximations, and that in higher-order approximations (starting, perhaps, from P-5) the situation is the reverse [1], was not confirmed by our calculations. Presumably one must choose a still higher order of approximation to have this statement satisfied.

2. Next we consider the albedo problem, i.e. the case where a delta function beam is incident at an angle whose cosine is u on an infinitely deep or finite plane-parallel slab.

In this case the equation of radiative transfer for determining the field of diffuse radiation is

$$\mu \frac{dI}{d\tau} + I(\tau, \mu) = \frac{1}{2} \int_{-1}^{+1} I(\tau, y)\, dy + \frac{1}{4} F e^{-\tau/u}, \tag{11}$$

where F is the intrinsic flux.

Moment equations of (11) are given as follows:

$$\begin{aligned} &\psi'_1(\tau) = \frac{1}{2} F e^{-\tau/u}, \\ &n\psi'_{n-1}(\tau) + (2n+1)\psi_n(\tau) + (n+1)\psi'_{n+1}(\tau) = 0, \quad n \geqslant 1. \end{aligned} \tag{12}$$

For a finite slab with the optical thickness of x the P-5 solution of equations (12) is the following:

$$\begin{aligned} \psi_0(\tau) &= A_1 + A_2\tau + \alpha + \beta + A_7 e^{-\tau/u}, \\ \psi_1(\tau) &= -\frac{1}{3} A_2 + B_7 e^{-\tau/u}, \\ \psi_2(\tau) &= -\frac{1}{2}(\alpha + \beta) + C_7 e^{-\tau/u}, \\ \psi_3(\tau) &= -\frac{5}{6}(\lambda_1^{-1}\alpha + \lambda_2^{-1}\beta) + D_7 e^{-\tau/u}, \\ \psi_4(\tau) &= p_1\alpha + p_2\beta + E_7 e^{-\tau/u}, \\ \psi_5(\tau) &= q_1\alpha + q_2\beta + G_7 e^{-\tau/u}, \end{aligned} \tag{13}$$

where

$$\alpha = A_3 e^{\lambda_1\tau} + A_4 e^{-\lambda_1\tau},$$
$$\beta = A_5 e^{\lambda_2\tau} + A_6 e^{-\lambda_2\tau},$$

$$A_7 = -\frac{3}{2} QFu^2(231 - 1190u^2 + 1155u^4),$$

$$B_7 = -\frac{1}{2} Fu,$$

$$C_7 = 3QFu^2(39 - 77u^2),$$

$$D_7 = QFu(25 - 99u^2),$$

$$E_7 = -44QFu^2,$$

$$G_7 = -20QFu$$

and

$$Q^{-1} = 75 - 882u^2 + 1155u^4.$$

It must be pointed out that this solution becomes indeterminate when

$$u = \lambda_1^{-1} = 0.3122126$$

or

$$u = \lambda_2^{-1} = 0.8161859.$$

For a semi-infinite medium the solution of equations (12) is also given by formulae (13), only

$$A_2 = A_3 = A_5 = 0$$

and the remaining coefficients can be determined by making use of boundary conditions (7). However, for a finite slab the set of restrictions (7) is not sufficient and it must be complemented by the following:

$$\psi_n(x) = \sum_{m=0}^{2} (-1)^m R_{6-n,m} \psi_m(x), \qquad n = 5,\ 4,\ 3. \tag{14}$$

This completes the formal solution of system (12) in the P-5 approximation.

As our aim is to compare the accuracy of various boundary conditions, we have to find the exact solution of equation (11) both for the cases of a semi-infinite and a finite slab.

For the finite slab the source function

$$S(u, \tau) = \frac{1}{2} \int_{-1}^{+1} I(\tau, y)\, dy + \frac{1}{4} F e^{-\tau/u} \tag{15}$$

satisfies the Fredholm integral equation

$$S(u,\tau)=\frac{1}{2}\int_0^{\infty} S(u,y)\,\mathrm{E}_1(|\tau-y|)\,dy+\frac{1}{4}F\,e^{-\tau/u}, \qquad (16)$$

where $\mathrm{E}_1(z)$ is the standard exponential integral of the first order.

One can solve equation (16) and find both the source function and the mean intensity. But, due to the fact that we are dealing with the case of pure scattering, the convergence of the solution of equation (16) is extremely slow, especially when considering the semi-infinite slab.

To solve equation (11) for this case we used the Ambartsumian integro-differential equation [5]

$$\frac{\partial}{\partial\tau}S(u,\tau)=-\frac{1}{u}S(u,\tau)+\frac{1}{2}H(u)\int_0^1 S(y,\tau)\frac{dy}{y}, \qquad (17)$$

where $H(u)$ is the Chandrasekhar H-function [6]. To produce the source function $S(u,\tau)$ we approximated the integral in equation (17) by Gaussian quadrature of order nine, normalized to the interval (0, 1).

This yields a system of nine ordinary differential equations, which we solved by the Runge-Kutta integration scheme, using the initial conditions

$$S(u_i,0)=\frac{1}{4}FH(u_i), \qquad i=1,\,2,\,\ldots,\,9.$$

To compute the Chandrasekhar H-function we used the Kourganoff-Busbridge formula [7]

$$H(z)=\exp\left\{-\frac{z}{\pi}\int_0^{\pi/2}\frac{\ln(1-y\operatorname{ctg}y)}{\cos^2 y+z^2\sin^2 y}\,dy\right\},$$

incorporating the brilliant technique of Stibbs and Weir [8]. The maximum relative error of H does not exceed 0.0002%. For the finite slab we used the approach of Bellman et al. [9]. The essence of their method is to solve the system for Chandrasekhar's X and Y functions [6]

$$\frac{\partial}{\partial x} X(x, u) = \frac{1}{2} Y(x, u) \int_0^1 Y(x, y) \frac{dy}{y},$$
$$\frac{\partial}{\partial x} Y(x, u) = -\frac{1}{u} Y(x, u) + \frac{1}{2} X(x, u) \int_0^1 Y(x, y) \frac{dy}{y} \tag{18}$$

from $x=0$ to $x=t$, using the boundary conditions

$$X(0, u) = 1$$

and

$$Y(0, u) = 1.$$

This yields us

$$W(t, t, u) = \frac{1}{4} FX(t, u),$$

which serves as a boundary condition for the following equation [5]

$$\frac{\partial}{\partial x} W(t, x, u) = -\frac{1}{u} W(t, x, u) + \frac{1}{2} X(x, u) \int_0^1 W(t, x, y) \frac{dy}{y} \tag{19}$$

where

$$W(t, x, u) = S(x - t, x, u).$$

Then equation (19) is adjoined to system (18) and the integration continues until x reaches the largest desired optical thickness.

The integrals in system (18) and equation (19) were treated in the manner described. According to the assertion of Bellman et al. [9], a sufficient accuracy was obtained even by making use of the seven-point quadrature formula.

The exact solutions both for the semi-infinite and the finite slabs ($x=0.1$) were compared with the P-5 solution and the relative error of the mean intensity ε is given in Tables 4 and 5. Three representative incident angles have been chosen,

$$\arccos u = 10^{0}.23731,$$
$$\arccos u = 52^{0}.25655,$$
$$\arccos u = 89^{0}.08782.$$

Table 4

Fractional errors of the mean intensity expressed as percentages. Albedo problem, semi-infinite medium

	$u=0.0159199$			$u=0.6121267$			$u=0.9840801$		
τ	MARK	MARSHAK	POMRANING	MARK	MARSHAK	POMRANING	MARK	MARSHAK	POMRANING
0	30.42	23.03	19.95	1.62	—2.36	—5.04	0.12	—2.69	—5.11
0.1	2.66	—4.51	—7.93	5.93	3.46	1.80	4.53	2.61	1.11
0.2	0.40	—4.93	—7.92	4.63	2.83	1.63	3.81	2.30	1.23
0.3	—1.31	—5.18	—7.83	3.38	1.98	1.04	2.96	1.71	0.89
0.4	—1.69	—4.45	—6.79	2.47	1.31	0.55	2.30	1.23	0.57
0.5	—1.66	—3.59	—5.71	1.85	0.86	0.21	1.84	0.89	0.34
0.6	—1.51	—2.89	—4.83	1.44	0.57	—0.00	1.51	0.65	0.18
0.7	—1.38	—2.37	—4.19	1.17	0.38	—0.14	1.29	0.50	—0.07
0.8	—1.27	—2.02	—3.76	0.99	0.26	—0.22	1.12	0.38	—0.00
0.9	—1.21	—1.81	—3.49	0.87	0.17	—0.28	1.00	0.30	—0.06
1.0	—1.19	—1.71	—3.34	0.77	0.10	—0.33	0.89	0.22	—0.11

Table 5

Fractional errors of the mean intensity expressed as percentages
Albedo problem. Finite slab ($x=0.1$)

	$u=0.0159199$			$u=0.6121267$			$u=0.9840801$		
τ	MARK	MARSHAK	POMRANING	MARK	MARSHAK	POMRANING	MARK	MARSHAK	POMRANING
0	43.99	32.50	27.79	29.11	13.74	7.30	28.88	13.45	6.98
0.01	43.69	32.54	27.94	34.06	20.32	14.55	33.82	20.02	14.22
0.02	40.11	28.37	23.49	36.26	23.38	17.95	36.07	23.13	17.68
0.03	35.64	22.97	17.67	37.24	24.81	19.57	37.10	24.64	19.38
0.04	31.23	17.51	11.75	37.58	25.37	20.21	37.52	25.28	20.12
0.05	27.29	12.52	6.29	37.55	25.37	20.22	37.56	25.37	20.23
0.06	23.93	8.14	1.44	37.21	24.90	19.70	37.29	24.99	19.80
0.07	21.11	4.31	—2.84	36.51	23.89	18.57	36.65	24.07	18.76
0.08	18.68	0.85	—6.75	35.22	22.06	16.52	35.42	22.31	16.79
0.09	16.40	—2.50	—10,59	32.80	18.73	12.81	33.04	19.03	13.14
0.10	13.97	—6.16	—14.78	27.91	12.20	5.60	28.14	12.49	5.92

The results of computation show that the Pomraning boundary conditions should be preferred if one is interested in the radiation field in deeper layers as well as in the case of normal incidence. This is applicable also to the solution in the P-1 or P-3 approximations.

The errors at the surface of semi-infinite slab are rather large when using the Pomraning boundary conditions. But the errors decrease rapidly towards the deeper layers.

For the finite slab the situation is to some extent inverted — the errors are smaller at boundaries and reach maximum near the centre of the slab. In this case the superiority of the Pomraning boundary conditions is obvious if only one is not dealing with large angles of incidence.

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Задача Милна и альбедная задача в P-5 приближении

Т. Вийк

Задача Милна и альбедная задача решены в P-5 приближении с использованием граничных условий различного типа.

Показано, что граничные условия Помрейнинга предпочтительнее, если рассматривать поле излучения в глубоких слоях среды и если угол падения излучения к среде не слишком большой.

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